Classification of Kac representations in the logarithmic minimal models $\mathcal{LM}(1,p)$

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Abstract

For each pair of positive integers r,s, there is a so-called Kac representation (r,s) associated with a Yang-Baxter integrable boundary condition in the lattice approach to the logarithmic minimal model $\mathcal{LM}(1,p)$. We propose a classification of these representations as finitely-generated submodules of Feigin-Fuchs modules, and present a conjecture for their fusion algebra which we call the Kac fusion algebra. The proposals are tested using a combination of the lattice approach and applications of the Nahm-Gaberdiel-Kausch algorithm. We also discuss how the fusion algebra may be extended by inclusion of the modules contragredient to the Kac representations, and determine polynomial fusion rings isomorphic to the conjectured Kac fusion algebra and its contragredient extension.

Contents

1 Introduction

The logarithmic minimal models $\mathcal{LM}(p,p')$ were introduced in [1], and the present paper concerns these models in the Virasoro picture without consideration for eventual extensions with respect to some $\mathcal{W}$-algebras. As logarithmic conformal field theories [2, 3, 4, 5], the models arise in the continuum scaling limit of an infinite family of Yang-Baxter integrable lattice models labelled by the pair of coprime integers p,p'. For each pair of positive integers $r,s\in\mathbb{N}$, there is a so-called Kac representation associated with an integrable boundary condition in the lattice model [6, 7]. Despite their importance, these Kac representations are in general rather poorly understood as modules over the Virasoro algebra. Their characters are known empirically from the lattice approach, but this is in general not sufficient to determine the underlying representations.

Fusion can be implemented on the lattice without detailed knowledge of the structure of the Kac representations. This gives significant insight into the fusion algebra generated from repeated

fusion of the Kac representations and has led to a concrete conjecture for the so-called fundamental fusion algebra and its representation content [6, 7]. This fundamental fusion algebra is generated from repeated fusion of the fundamental Kac representations $(2,1)$ and $(1,2)$, but does not involve all Kac representations. The lattice implementation of fusion also provides further information on the structure of the representations themselves, but certain crucial questions are left unanswered. However, the fusion rules should be compatible with the outcome of the Nahm-Gaberdiel-Kausch (NGK) algorithm [8, 9], thus providing an additional tool to determine the fusion rules in $\mathcal{LM}(p,p')$. The NGK algorithm has played a prominent role in the study of fusion in the so-called augmented $c_{p,p'}$ models [9, 10] as well as in [11, 12] on critical percolation and related models. Alternative approaches to the computation of fusion rules in these models are discussed in [13, 14].

The logarithmic minimal models are non-rational conformal field theories as they contain infinitely many Virasoro representations. Some of these can be organized in finitely many extended representations associated with new integrable boundary conditions [15, 16, 17]. This is referred to as the $\mathcal{W}$-extended picture of the logarithmic minimal models, and the extension is believed to be with respect to the triplet $\mathcal{W}_{p,p'}$ algebra [18, 19, 20]. Due to their 'rational nature', these $\mathcal{W}$-extended models have been studied extensively, see [21] and references therein.

Here we consider the infinite family of logarithmic minimal models $\mathcal{LM}(1,p)$. We propose a classification of the Kac representations (r,s) for all $r,s\in\mathbb{N}$ as finitely-generated submodules of Feigin-Fuchs modules [22], and present a conjecture for their fusion algebra. We thus find that the only higher-rank representations generated by repeated fusion of the Kac representations are the rank-2 modules $\mathcal{R}_r^b$ already present in the fundamental fusion algebra. The proposals are tested using a combination of the lattice approach and applications of the NGK algorithm. Under some natural assumptions about the continuum scaling limit of the lattice model, some results are in fact *exact* rather than conjectural. We also discuss how the fusion algebra may be extended by inclusion of the modules contragredient to the Kac representations, and determine polynomial fusion rings isomorphic to the conjectured Kac fusion algebra and its contragredient extension.

2 Logarithmic minimal model $\mathcal{LM}(1,p)$

The logarithmic minimal model $\mathcal{LM}(1,p)$ is a logarithmic conformal field theory with central charge

$$c = 1-6\frac{(p-1)^2}{p} \tag{2.1}$$

Here we are mainly interested in the Virasoro representations associated with the boundary conditions appearing in the lattice approach to $\mathcal{LM}(1,p)$ as described in [1, 7], but consider also other representations.

2.1 Highest-weight modules

Before describing the representations associated with boundary conditions, let us recall some basic facts about highest-weight modules over the Virasoro algebra with central charge given by (2.1). For each pair of positive Kac labels $r,s\in\mathbb{N}$, the highest-weight Verma module of conformal weight $\Delta_{r,s}$ is denoted by $V_{r,s}$ where $\Delta_{r,s}$ is given by the Kac formula

$$\Delta_{r,s} = \frac{(rp-s)^2-(p-1)^2}{4p}, \qquad r,s\in\mathbb{Z} \tag{2.2}$$

As indicated, it is convenient to consider also negative or vanishing Kac labels, in particular when applying the Kac-table symmetries

$$\Delta_{r,s} = \Delta_{-r,-s}, \qquad \Delta_{r,s} = \Delta_{r+k,s+kp}, \qquad k \in \mathbb{Z} \tag{2.3}$$

The distinct conformal weights appearing in (2.2) appear exactly once in the set

$$\{\Delta_{r,s};\, r \in \mathbb{N},\, s \in \mathbb{Z}_{1,p}\} \tag{2.4}$$

where we have introduced the notation

$$\mathbb{Z}_{n,m} = \mathbb{Z} \cap [n, m] \tag{2.5}$$

The Verma module $V_{r,s}$ has a proper submodule at Virasoro level rs given by $V_{r,-s}$ (where $V_{r,-s} = V_{r',s'}$ for some $r', s' \in \mathbb{N}$), allowing us to define the quotient module

$$Q_{r,s} = V_{r,s}/V_{r,-s} \tag{2.6}$$

Its character is given by

$$\chi[Q_{r,s}](q) = \frac{q^{\frac{1-c}{24}+\Delta_{r,s}}}{\eta(q)}\big(1 - q^{rs}\big) = \frac{q^{(rp-s)^2/4p}}{\eta(q)}\big(1 - q^{rs}\big) \tag{2.7}$$

where q is the modular nome while the Dedekind eta function is given by

$$\eta(q) = q^{1/24} \prod_{m\in\mathbb{N}} (1 - q^m) \tag{2.8}$$

This module is in general not irreducible. The irreducible highest-weight module $M_{r,s}$ of conformal weight $\Delta_{r,s}$ is obtained by quotienting out the *maximal* proper submodule of $V_{r,s}$, and we denote its character by

$$\mathrm{ch}_{r,s}(q) = \chi[M_{r,s}](q) \tag{2.9}$$

For

$$s = s_0 + kp, \qquad s_0 \in \mathbb{Z}_{1,p-1}; \quad k \in \mathbb{N}_0 \tag{2.10}$$

the character of the quotient module $Q_{r,s}$ can be written as

$$\chi[Q_{r,s}](q) = \sum_{j=0}^{\min(2r-1,2k)} \mathrm{ch}_{r+k-j,(-1)^j s_0+(1-(-1)^j)p/2}(q) \tag{2.11}$$

2.2 Kac representations

There is a so-called Kac representation (r, s) for each pair of positive Kac labels $r, s \in \mathbb{N}$. It is associated with a Yang-Baxter integrable boundary condition in the lattice approach to $\mathcal{LM}(1, p)$ [1, 7] and arises in the continuum scaling limit. As we will discuss, these Kac representations can be irreducible, fully reducible or reducible yet indecomposable as modules over the Virasoro algebra. They are all of rank 1 as the dilatation generator (the Virasoro mode L_0) is found to be diagonalizable. Empirically, the Virasoro character of the Kac representation (r, s) is identical to the character (2.7) of the quotient module $Q_{r,s}$

$$\chi_{r,s}(q) = \chi[Q_{r,s}](q) \tag{2.12}$$

Aside from its character and its rank-1 nature, it is not, however, a priori clear from the lattice what type of Virasoro module the Kac representation (r, s) actually is. A typical dilemma is the distinction between a reducible yet indecomposable module and the direct sum of its irreducible subfactors (subquotients). By construction, the indecomposable module has the same character as the direct sum but they are nevertheless inequivalent due to the indecomposable nature of the former. The situation can be rather intricate, as we will argue below, since some Kac representations are found to be non-highest-weight representations. Our assertion is that they can all be viewed as finitely-generated submodules of Feigin-Fuchs modules, see (2.28) and (2.34), for example. In particular, despite the character identity (2.12), we thus assert that (r, s) and $Q_{r,s}$ in general differ as representations.

It follows from the character expressions that (r, s) is irreducible for $s \in \mathbb{Z}_{1,p}$ and that $(1, kp)$ is irreducible for $k \in \mathbb{N}$, thus giving rise to the identifications $(1, rp) \equiv (r, p)$ [7]. These are the only irreducible Kac representations. Following Section 4.2 in [7], one deduces that the Kac representation (r, kp) is fully reducible

$$(r, kp) = \bigoplus_{j=|r-k|+1,\,\text{by}\,2}^{r+k-1} (j, p) \tag{2.13}$$

The remaining Kac representations were not fully characterized in [7]. Below, we offer a conjecture for the classification of the *full* set of Kac representations.

2.3 Rank-2 representations

The infinite family

$$\{\mathcal{R}_r^b\,;\, r \in \mathbb{N},\, b \in \mathbb{Z}_{1,p-1}\} \tag{2.14}$$

of reducible yet indecomposable modules of rank 2 arises from repeated fusion of *irreducible* Kac representations [7]. This follows by isolating $\mathcal{R}_r^b$ in

$$(1, b+1) \otimes (1, rp) = \bigoplus_{\beta}^{b} \mathcal{R}_r^{\beta}, \qquad b \in \mathbb{Z}_{1,p-1} \tag{2.15}$$

for example, as b increases from 1 to $p-1$. Here we have introduced the summation convention

$$\bigoplus_{n}^{N} R_n = \bigoplus_{n=\epsilon(N),\,\text{by}\,2}^{N} R_n, \qquad \epsilon(N) = \frac{1-(-1)^N}{2} = N \ (\text{mod } 2) \tag{2.16}$$

and extended the notation $\mathcal{R}_r^b$ by writing

$$\mathcal{R}_r^0 \equiv (1, rp) \equiv (r, p) \tag{2.17}$$

for the irreducible rank-1 module (r, p). The rank-2 module $\mathcal{R}_r^b$ is characterized by the structure diagram

$$\begin{array}{ccc} & M_{2,b} & \\ \swarrow & & \nwarrow \\ M_{1,p-b} & \longleftarrow & M_{1,p-b} \end{array} \;, \qquad \mathcal{R}_r^b: \quad \begin{array}{ccc} & M_{r+1,b} & \\ \swarrow & & \nwarrow \\ M_{r,p-b} & \longleftarrow & M_{r,p-b} \\ \nwarrow & & \swarrow \\ & M_{r-1,b} & \end{array} \;, \qquad r \in \mathbb{Z}_{\geq 2} \tag{2.18}$$

where the first diagram is $\mathcal{R}_1^b:$

where an arrow from the irreducible subfactor M to the irreducible subfactor M' indicates that vectors in M are mapped not only to vectors in M itself but also to vectors in M' by the action of the Virasoro

algebra. An arrow from one copy of M to another copy of M indicates that L_0 is non-diagonalizable and that the module is of rank 2. Representations of rank $\rho > 2$ are not present here, but would otherwise require ρ copies of a given irreducible subfactor suitably arranged in a chain and connected by aligned arrows. The character of the rank-2 module $\mathcal{R}_r^b$ follows from the structure diagram (2.18) and is given by

$$\chi[\mathcal{R}_r^b](q) = (1-\delta_{r,1})\mathrm{ch}_{r-1,b}(q) + 2\mathrm{ch}_{r,p-b}(q) + \mathrm{ch}_{r+1,b}(q) \tag{2.19}$$

According to the fusion algebra conjectured in Section 3.2, no additional rank-2 modules nor higher-rank modules are generated from repeated fusion of the *full* set of Kac representations (r,s).

2.4 Reducible yet indecomposable Kac representations

There is a pair of Feigin-Fuchs modules corresponding to each Verma module $V_{r,s}$. We denote them by $F_{r,s}^{\rightarrow}$ and $F_{r,s}^{\leftarrow}$, and they can be constructed by reversing every second arrow in the structure diagram for $V_{r,s}$. The arrow on $F_{r,s}^{\rightarrow}$ indicates that vectors in $M_{r,s}$ are mapped not only to vectors in $M_{r,s}$ itself but also *to* vectors in the next subfactor by the action of the Virasoro algebra. Similarly, the arrow on $F_{r,s}^{\leftarrow}$ indicates that vectors in $M_{r,s}$ can be reached *from* vectors in the next subfactor. Likewise, we can associate a pair of finitely-generated Feigin-Fuchs modules to every quotient module $Q_{r,s}$. For $2r-1 < 2k$, where $s = s_0 + kp$, the Feigin-Fuchs modules corresponding to $Q_{r,s}$ are characterized by the structure diagrams

$$\begin{aligned} Q_{r,s}^{\rightarrow}: &\qquad M_{k-r+1,p-s_0} \rightarrow M_{k-r+2,s_0} \leftarrow M_{k-r+3,p-s_0} \rightarrow \ldots \leftarrow M_{k+r-1,p-s_0} \rightarrow M_{k+r,s_0} \\ Q_{r,s}^{\leftarrow}: &\qquad M_{k-r+1,p-s_0} \leftarrow M_{k-r+2,s_0} \rightarrow M_{k-r+3,p-s_0} \leftarrow \ldots \rightarrow M_{k+r-1,p-s_0} \leftarrow M_{k+r,s_0} \end{aligned} \tag{2.20}$$

For $2r-1 > 2k$, the Feigin-Fuchs modules corresponding to $Q_{r,s}$ are characterized by the structure diagrams

$$\begin{aligned} Q_{r,s}^{\rightarrow}: &\qquad M_{r-k,s_0} \rightarrow M_{r-k+1,p-s_0} \leftarrow M_{r-k+2,s_0} \rightarrow \ldots \rightarrow M_{r+k-1,p-s_0} \leftarrow M_{r+k,s_0} \\ Q_{r,s}^{\leftarrow}: &\qquad M_{r-k,s_0} \leftarrow M_{r-k+1,p-s_0} \rightarrow M_{r-k+2,s_0} \leftarrow \ldots \leftarrow M_{r+k-1,p-s_0} \rightarrow M_{r+k,s_0} \end{aligned} \tag{2.21}$$

By construction, we have

$$\chi[Q_{r,s}^{\rightarrow}](q) = \chi[Q_{r,s}^{\leftarrow}](q) = \chi[Q_{r,s}](q) \tag{2.22}$$

In all cases, the Feigin-Fuchs modules $Q_{r,s}^{\rightarrow}$ and $Q_{r,s}^{\leftarrow}$ are *contragredient* to each other where the contragredient module A^* to a module A is obtained by reversing all structure arrows between its irreducible subfactors. It follows that $\chi[A^*](q) = \chi[A](q)$ and that $A^{**} = A$.

Since we know the structure of the Kac representation (r,s) for $s \le p$ (irreducible) or $s = kp$ (fully reducible), we now consider the cases where s is of the form (2.10) for $k \ge 1$.

Highest-weight assumption. The Kac representation $(1, s_0+kp)$ is the indecomposable highest-weight module $Q_{1,s_0+kp}^{\rightarrow}$, that is,

$$(1, s_0+kp) = Q_{1,s_0+kp}^{\rightarrow} = Q_{1,s_0+kp} = V_{1,s_0+kp}/V_{1,s_0+(k+2)p}: \qquad M_{1,s_0+kp} \rightarrow M_{1,(k+2)p-s_0} \tag{2.23}$$

It is emphasized that, a priori, this Kac representation could be the direct sum of its two irreducible subfactors or contragredient to the highest-weight module (2.23). Consistency of the fusion algebra excludes the first possibility, though. To appreciate this, let us initially compare certain fusion properties

of the Kac representation $(1,p+1)$ with the similar properties of the direct sum $(1,p-1)\oplus(2,1)$ of its constituent subfactors. According to the fundamental fusion algebra [7], we have

$$\big[(1,p-1)\oplus(2,1)\big]\otimes\big[(1,p-1)\oplus(2,1)\big]=2(1,1)\oplus(3,1)\oplus 2(2,p-1)\oplus\bigoplus_{\beta}^{p-3}\mathcal{R}_1^{\beta} \qquad (2.24)$$

On the other hand, it is observed from the lattice that the decomposition of the fusion product $(1,p+1)\otimes(1,p+1)$ for small p contains rank-2 Jordan cells linking the two copies of the irreducible subfactor $M_{1,1}=(1,1)$. This is incompatible with (2.24) as the former indicates the presence of the rank-2 module $\mathcal{R}_1^{p-1}$ (2.18) in the decomposition, in accordance with the conjectured fusion rule (3.7)

$$(1,p+1)\otimes(1,p+1)=(1,2p+1)\oplus\bigoplus_{\beta}^{p-1}\mathcal{R}_1^{\beta} \qquad (2.25)$$

More generally, the lattice approach provides similar evidence for the indecomposability of $(1,s_0+kp)$. We thus find that

$$(1,s_0+kp)\neq(k,p-s_0)\oplus(k+1,s_0) \qquad (2.26)$$

since the lattice approach indicates the presence of $\mathcal{R}_1^{p-1}$ in the decomposition of the fusion product $(1,s_0+kp)\otimes(1,s_0+kp)$, while the decomposition of the fusion product of $(k,p-s_0)\oplus(k+1,s_0)$ with itself follows from the fundamental fusion algebra

$$\big[(k,p-s_0)\oplus(k+1,s_0)\big]\otimes\big[(k,p-s_0)\oplus(k+1,s_0)\big]=2(1,1)\oplus\ldots \qquad (2.27)$$

and contains two *unlinked* copies of $M_{1,1}$.

We are still faced with the problem of identifying the Kac representations $(1,s_0+kp)$ as highest-weight modules or as the corresponding contragredient modules. As indicated, here we *assume* that they are highest-weight modules and then study the implications of this assumption. We will nevertheless return to this question in Section 2.5 and Section 3.

Structure conjecture. For $s=s_0+kp$, $k\in\mathbb{N}$, the Kac representation (r,s) is the Feigin-Fuchs module

$$(r,s)=\begin{cases}Q_{r,s}^{\rightarrow}, & 2r-1<2k\\[4pt] Q_{r,s}^{\leftarrow}, & 2r-1>2k\end{cases} \qquad (2.28)$$

Below, we present arguments in support of this conjecture by combining results from the lattice approach with results from applications of the NGK algorithm based on (2.23).

The range for s_0 in (2.28) can be extended from $\mathbb{Z}_{1,p-1}$ to $\mathbb{Z}_{0,p-1}$ such that s can be any positive integer $s\in\mathbb{N}$ (where we exclude $s_0=k=0$ for which $s=0$). For $s_0=0$, the structure diagrams associated with $Q_{r,s}^{\rightarrow}$ and $Q_{r,s}^{\leftarrow}$ in (2.28) are separable (degenerate) and the modules are fully reducible

$$Q_{r,kp}^{\rightarrow}=Q_{r,kp}^{\leftarrow}=Q_{r,kp}=\bigoplus_{j=|r-k|+1,\,\mathrm{by}\,2}^{r+k-1}M_{j,p} \qquad (2.29)$$

in accordance with (2.13). It is noted that this decomposition is symmetric in r and k.

In retrospect, we could have *defined* a Kac representation (r,s) mathematically, for general $r,s\in\mathbb{N}$, as the finitely-generated Feigin-Fuchs submodule (2.28) where

$$s=s_0+kp,\qquad s_0\in\mathbb{Z}_{0,p-1},\quad k\in\mathbb{N}_0 \qquad (2.30)$$

From the lattice, we would then conjecture that the Virasoro modules associated with the aforementioned boundary conditions are Kac representations in the mathematical sense just given. A major goal of the present work is indeed to collect evidence for this conjecture. It is recalled that we are working under the assumption that the modules $(1, s_0 + kp)$ are highest-weight modules.

2.4.1 Evidence for the structure conjecture

Fusion can be implemented on the lattice without detailed knowledge of the structure of the Kac representations. The Kac representation (r, s) itself is actually constructed by fusing the 'horizontal' Kac representation $(r, 1)$ with the 'vertical' Kac representation $(1, s)$

$$(r, s) = (r, 1) \otimes (1, s) \tag{2.31}$$

Under the assumption (2.23), we have applied the NGK algorithm to many fusion products of this kind and they all corroborate the structure conjecture (2.28). Some of our findings and observations are summarized in the following with additional details deferred to Appendix A.

In the decomposition of a fusion product examined using the NGK algorithm, the vectors appearing at Nahm level 0 are the ones which are not the image of negative Virasoro modes. These vectors[1] constitute the minimal set of vectors from which the entire (decomposable or indecomposable) module, arising as the result of the fusion product, can be generated by the action of negative Virasoro modes only. It thus suffices to analyze a fusion product at Nahm level 0 in order to identify this minimal set of vectors. This knowledge is then sufficient to distinguish between a highest-weight module like (2.23) and its contragredient module. Indeed, the minimal set of vectors associated with the highest-weight module in (2.23) consists of only one vector, namely $|\Delta_{1,s_0+kp}\rangle$, whereas the minimal set associated with the contragredient module consists of the two vectors $|\Delta_{1,s_0+kp}\rangle$ and $|\Delta_{1,(k+2)p-s_0}\rangle$.

Once we know this minimal set, we can use our knowledge of the character $\chi_{r,s}(q)$ to deduce the number and conformal weights of the vectors appearing at higher Nahm levels. This is very helpful when determining the otherwise evasive spurious subspaces appearing in the NGK algorithm, see Appendix A.

Singular vector conjecture. With the normalization convention for singular vectors used in Appendix A.2, we conjecture that at Nahm level 0 in the fusion product $(2, 1) \otimes (1, s)$

$$|\Delta_{2,1}\rangle \times |\lambda_{1,s}\rangle = -\Big(\prod_{j=1}^{s-1} \frac{(p+j)(p-j)}{p}\Big)\big\{L_{-1} \times I + \tfrac{s-1}{2} I \times I\big\}|\Delta_{2,1}\rangle \times |\Delta_{1,s}\rangle \tag{2.32}$$

We have verified this remarkably simple expression explicitly for $s \leq 6$. The action of the co-multiplication of L_0 on the corresponding two-dimensional initial vector space is given by

$$\begin{aligned} \Delta(L_0)|\Delta_{2,1}\rangle \times |\Delta_{1,s}\rangle &= (\Delta_{2,1} + \Delta_{1,s})|\Delta_{2,1}\rangle \times |\Delta_{1,s}\rangle + L_{-1}|\Delta_{2,1}\rangle \times |\Delta_{1,s}\rangle \\ \Delta(L_0)L_{-1}|\Delta_{2,1}\rangle \times |\Delta_{1,s}\rangle &= p\Delta_{1,s}|\Delta_{2,1}\rangle \times |\Delta_{1,s}\rangle + (\Delta_{2,1} + \Delta_{1,s} + 1 - p)L_{-1}|\Delta_{2,1}\rangle \times |\Delta_{1,s}\rangle \end{aligned} \tag{2.33}$$

It follows readily from (2.32) that a spurious subspace at Nahm level 0 is generated by setting the singular vector $|\lambda_{1,s}\rangle = 0$ if and only if $s \leq p$, in which case this subspace is one-dimensional. For $s \leq p$, the matrix realization of $\Delta(L_0)$ is therefore one-dimensional and is given by $\Delta_{2,s}$, reflecting that the Kac representation $(2, s)$ is irreducible for all $s \leq p$. For $s > p$, it follows from (2.33) that the two-dimensional matrix realization of $\Delta(L_0)$ is diagonalizable and has eigenvalues $\Delta_{1,s-p}$ and $\Delta_{1,s+p}$.

[1]These vectors actually span a subspace, but it is convenient to think in terms of a set of basis vectors.

For $s = kp$, this is in accordance with the decomposition (2.13), while for $s = s_0 + kp$, it is in accordance with the structure conjecture (2.28).

At Nahm level 0, we have confirmed the structure conjecture (2.28) in many cases. In some of these, we have continued the analysis to higher Nahm level and always with affirmative results. Details of the analysis for $(2,3)$ in critical dense polymers $\mathcal{LM}(1,2)$ appear in Appendix A.4 and are summarized by the structure diagram

$$(2,3) = Q^{\leftarrow}_{2,3} : \qquad \mathcal{V}(0) \leftarrow \mathcal{V}(1) \rightarrow \mathcal{V}(3) \tag{2.34}$$

where $\mathcal{V}(\Delta)$ denotes the irreducible highest-weight module of conformal weight Δ.

2.5 Contragredient Kac representations

We recall our working assumption (2.23) that the reducible yet indecomposable Kac representation $(1, s_0 + kp)$ is the highest-weight module $Q^{\rightarrow}_{1,s_0+kp}$ and not its contragredient module $Q^{\leftarrow}_{1,s_0+kp}$. It then follows from the structure diagram (2.18) that the rank-2 module $\mathcal{R}^b_r$ admits the short exact sequence

$$0 \rightarrow (1, rp - b) \rightarrow \mathcal{R}^b_r \rightarrow (1, rp + b) \rightarrow 0 \tag{2.35}$$

It also admits the short exact sequence

$$0 \rightarrow (r, p - b) \rightarrow \mathcal{R}^b_r \rightarrow (r, p + b) \rightarrow 0 \tag{2.36}$$

in terms of the (for $r \neq 1$) reducible yet indecomposable non-highest-weight module $(r, p + b)$.

As Virasoro modules, the Feigin-Fuchs modules contragredient to the ones appearing in (2.28), namely

$$(r,s)^* = \begin{cases} Q^{\leftarrow}_{r,s}, & 2r - 1 < 2k \\ Q^{\rightarrow}_{r,s}, & 2r - 1 > 2k \end{cases} \tag{2.37}$$

are perfectly well defined. An immediate application is to provide alternative characterizations of the rank-2 module $\mathcal{R}^b_r$ in terms of short exact sequences as we have

$$\begin{gathered} 0 \rightarrow (1, rp + b)^* \rightarrow \mathcal{R}^b_r \rightarrow (1, rp - b)^* \rightarrow 0 \\ 0 \rightarrow (r, p + b)^* \rightarrow \mathcal{R}^b_r \rightarrow (r, p - b)^* \rightarrow 0 \end{gathered} \tag{2.38}$$

noting that the rank-2 modules are invariant under reversal of structure arrows

$$(\mathcal{R}^b_r)^* = \mathcal{R}^b_r \tag{2.39}$$

For $r > 1$, the rank-2 module $\mathcal{R}^b_r$ thus admits four independent non-trivial short exact sequences. This is in accordance with the structure diagram (2.18) for $\mathcal{R}^b_r$ as it follows from the diagram that $\mathcal{R}^b_r$ has four inequivalent proper submodules. We also note that a fully reducible (in particular irreducible) module is identical to its contragredient module

$$(r,s)^* = (r,s), \qquad r \in \mathbb{N}; \quad s \in \mathbb{Z}_{1,p-1} \cup p\mathbb{N} \tag{2.40}$$

It seems natural to expect that the category or family of representations appearing in the full-fledged logarithmic conformal field theory $\mathcal{LM}(1,p)$ is closed under reversal of structure arrows in the sense that the contragredient module to a module in the category is also in the category. Above, we have only considered the Virasoro modules associated with boundary conditions [1, 7], namely the Kac

representations (r,s) and the rank-2 modules $\mathcal{R}_r^b$. As already mentioned, these rank-2 modules are invariant under reversal of structure arrows, whereas the only invariant Kac representations are the fully reducible ones (including the irreducible ones). As a consequence of the indicated expectation, the *contragredient Kac representations* (2.37) should also be members of the invariant category. This situation resembles the logarithmic minimal models $\mathcal{LM}(p,p')$ in the so-called $\mathcal{W}$-extended picture [15, 17] in which the modules associated with boundary conditions only constitute a subcategory of the full category if $p>1$. This idea was originally put forward and examined in [23] and has since been studied in more detail [24, 25, 26, 27]. In Section 3.3, we discuss how the fusion algebra generated by the Kac representations may be extended by the inclusion of the contragredient Kac representations.

3 Fusion algebras

3.1 Fundamental fusion algebra

There are infinitely many fusion (sub)algebras associated with $\mathcal{LM}(1,p)$. The *fundamental fusion algebra* [7]

$$\big\langle(1,1),(2,1),(1,2)\big\rangle \tag{3.1}$$

in particular, is generated from the two fundamental Kac representations $(2,1)$ and $(1,2)$ in addition to the identity $(1,1)$. This fusion algebra involves all the irreducible Kac representations and all the rank-2 representations (2.14). On the other hand, no reducible yet indecomposable Kac representations arise as the result of repeated fusion of the fundamental Kac representations. The fundamental fusion algebra has two canonical subalgebras

$$\big\langle(1,1),(2,1)\big\rangle,\qquad \big\langle(1,1),(1,2)\big\rangle \tag{3.2}$$

3.2 Kac fusion algebra

The *Kac fusion algebra* is generated by the *full* set of Kac representations

$$\big\langle(r,s);\ r,s\in\mathbb{N}\big\rangle \tag{3.3}$$

and its description is a main objective of this work. To appreciate this fusion algebra, it is instructive to examine its vertical component

$$\big\langle(1,s);\ s\in\mathbb{N}\big\rangle \tag{3.4}$$

which is characterized by the fusion rules of the vertical component $\langle(1,1),(1,2)\rangle$ of the fundamental fusion algebra supplemented by the fusion rules involving the reducible yet indecomposable Kac representations $(1,s_0+kp)$. To describe (3.4), we introduce the sign function

$$\mathrm{sg}(n)=\begin{cases}1, & n>0\\ -1, & n<0\end{cases} \tag{3.5}$$

Since this function only appears in conjunction with certain constraints, the value $\mathrm{sg}(0)$ turns out to be immaterial.

Fusion conjecture. The vertical component of the Kac fusion algebra satisfies

$$\big\langle(1,s);\ s\in\mathbb{N}\big\rangle=\big\langle(1,b+kp),\mathcal{R}_r^b;\ b\in\mathbb{Z}_{0,p-1},\ k\in\mathbb{N}_0,\ r\in\mathbb{N}\big\rangle \tag{3.6}$$

where we recall $\mathcal{R}_r^0 \equiv (1, rp)$ and set $(1,0) \equiv \mathcal{R}_0^\beta \equiv 0$, and is characterized by the fusion rules[2]

$$
\begin{aligned}
(1, b+kp) \otimes (1, b'+k'p) \;&=\; \bigoplus_{j=|k-k'|+1,\,\text{by }2}^{k+k'-1} \; \bigoplus_{\beta}^{p-|b-b'|-1} \mathcal{R}_j^\beta \oplus \bigoplus_{j=|k-k'+\text{sg}(b-b')|+1,\,\text{by }2}^{k+k'} \; \bigoplus_{\beta}^{|b-b'|-1} \mathcal{R}_j^\beta \\
&\oplus \bigoplus_{\beta}^{b+b'-p-1} \mathcal{R}_{k+k'+1}^\beta \oplus \bigoplus_{\beta=|b-b'|+1,\,\text{by }2}^{p-|p-b-b'|-1} (1, \beta + (k+k')p) \\
\mathcal{R}_r^b \otimes (1, b'+k'p) \;&=\; \Big(\bigoplus_{j=|r-k'|+1,\,\text{by }2}^{r+k'-1} \; \bigoplus_{\beta}^{p-|b-b'|-1} \mathcal{R}_j^\beta \oplus \bigoplus_{j=|r-k'-1+\text{sg}(p-b-b')|+1,\,\text{by }2}^{r+k'-\text{sg}(p-b-b')} \; \bigoplus_{\beta}^{|p-b-b'|-1} \mathcal{R}_j^\beta \\
&\oplus \bigoplus_{j=|r-k'-1|+1,\,\text{by }2}^{r+k'-2} \; \bigoplus_{\beta}^{p-|p-b-b'|-1} \mathcal{R}_j^\beta \oplus \bigoplus_{j=|r-k'+\text{sg}(b-b')|+1,\,\text{by }2}^{r+k'} \; \bigoplus_{\beta}^{|b-b'|-1} \mathcal{R}_j^\beta \\
&\oplus \bigoplus_{\beta}^{b'-b-1} \mathcal{R}_{r+k'}^\beta \oplus \bigoplus_{\beta=|b-b'|+1,\,\text{by }2}^{p-|p-b-b'|-1} \mathcal{R}_{r+k'}^\beta \Big) / (1+\delta_{b,0}) \\
\mathcal{R}_r^b \otimes \mathcal{R}_{r'}^{b'} \;&=\; \Big(\bigoplus_{j=|r-r'|,\,\text{by }2}^{r+r'} \big(2 - \delta_{j,|r-r'|}\big) \Big\{ \bigoplus_{\beta}^{|b-b'|-1} \oplus \big(1 - \delta_{j,r+r'}\big) \bigoplus_{\beta}^{p-|p-b-b'|-1} \Big\} \mathcal{R}_j^\beta \\
&\oplus \Big\{ \bigoplus_{j=|r-r'-1+\text{sg}(p-b-b')|+1,\,\text{by }2}^{r+r'-\text{sg}(p-b-b')} \oplus \bigoplus_{j=|r-r'+1-\text{sg}(p-b-b')|+1,\,\text{by }2}^{r+r'-1} \Big\} \bigoplus_{\beta}^{|p-b-b'|-1} \mathcal{R}_j^\beta \\
&\oplus \bigoplus_{\beta=|b-b'|+1,\,\text{by }2}^{p-|p-b-b'|-1} \mathcal{R}_{r+r'}^\beta \oplus \bigoplus_{\beta=|p-b-b'|+1,\,\text{by }2}^{p-|b-b'|-1} \mathcal{R}_{r+r'-1}^\beta \oplus \bigoplus_{\beta}^{p-b-b'-1} \mathcal{R}_{r+r'-1}^\beta \\
&\oplus \bigoplus_{j=|r-r'|+1,\,\text{by }2}^{r+r'-1} \big(2 - \delta_{j,r+r'-1}\big) \bigoplus_{\beta}^{p-|b-b'|-1} \mathcal{R}_j^\beta \Big) / \{(1+\delta_{b,0})(1+\delta_{b',0})\}
\end{aligned}
\tag{3.7}
$$

The divisions by $(1+\delta_{b,0})$ and $(1+\delta_{b',0})$ ensure that the fusion rules for $\mathcal{R}_r^0$ match those for $(1, rp)$. Evidence for this fusion conjecture is presented in Section 3.2.2 and Section 3.2.3.

Mnemonically, the fusion rules (3.7) are reconstructed straightforwardly using the underlying $sl(2)$ structure [7]. This structure is evident from the lattice where defects can be annihilated in pairs thus implying that the fusion product of two Kac representations $(1,s)$ and $(1,s')$ can be decomposed, up to indecomposable structures, as a sum of Kac representations

$$
(1,s) \otimes (1,s') = (1, |s-s'|+1) \overset{?}{\oplus} (1, |s-s'|+3) \overset{?}{\oplus} \ldots \overset{?}{\oplus} (1, s+s'-1)
\tag{3.8}
$$

The question marks indicate that the sums can be *direct* or *indecomposable*. The $sl(2)$ structure of the

[2]This revises the conjecture in [28] for the decomposition of the fusion product $(1, 2j_1-1) \otimes (1, 2j_2-1)$ in $\mathcal{LM}(1,2)$, see also Appendix B.

fusion product $(1,s)\otimes(1,s')$ is thus encoded in the character decomposition

$$\begin{aligned}
\chi\big[(1,s)\otimes(1,s')\big](q) &= \chi\big[(1,|s-s'|+1)\oplus(1,|s-s'|+3)\oplus\ldots\oplus(1,s+s'-1)\big](q) \\
&= \sum_{s''=|s-s'|+1,\ \mathrm{by}\ 2}^{s+s'-1} \chi_{1,s''}(q) \\
&= \sum_{t=0}^{\min\{s,s'\}-1} \chi_{1,s+s'-2t-1}(q)
\end{aligned} \tag{3.9}$$

Following the discussion of short exact sequences in Section 2.5, we may view the rank-2 module $\mathcal{R}_r^b$ as an indecomposable combination of the two Kac representations $(1,rp-b)$ and $(1,rp+b)$, that is,

$$\mathcal{R}_r^b = (1,rp-b)\oplus_i(1,rp+b) \tag{3.10}$$

Utilizing this, we introduce the 'forgetful functor' $\mathcal{F}$ by

$$\mathcal{F}[(1,s)] = (1,s), \qquad \mathcal{F}[\mathcal{R}_r^b] = (1,rp-b)\oplus(1,rp+b), \qquad \mathcal{F}[\mathcal{A}\otimes\mathcal{B}] = \mathcal{F}[\mathcal{F}[\mathcal{A}]\otimes\mathcal{F}[\mathcal{B}]] \tag{3.11}$$

and apply it to the various fusion products such as

$$\mathcal{F}[(1,s)\otimes(1,s')] = \bigoplus_{s''=|s-s'|+1,\ \mathrm{by}\ 2}^{s+s'-1} (1,s'') \tag{3.12}$$

We note that applying $\mathcal{F}$ does not correspond to moving to the Grothendieck ring associated with characters since we are working here with the reducible yet indecomposable Kac representations $(1,rp\pm b)$. Clearly, $\mathcal{F}$ does not have an inverse, but on fusion products, we can devise a prescription that 'reintroduces' the rank-2 modules in a unique and well-defined way. To describe this prescription, let us consider the fusion product $(1,s)\otimes(1,s')$ in (3.12) and initially focus on the Kac representation $(1,s_1'')$ with minimal Kac label $s_1''=|s-s'|+1$. Depending on p, this will appear as the submodule $(1,rp-b)$ of the rank-2 module $\mathcal{R}_r^b$ if and only if the matching module $(1,rp+b)$ also appears in the decomposition in (3.12). If not, the Kac representation $(1,s_1'')$ will appear 'by itself' in the decomposition of the fusion product. Having completed the examination of $(1,s_1'')$, we remove it together with its potential partner $(1,rp+b)$ from the direct sum in (3.12) and repeat the analysis for $(1,s_2'')$ corresponding to the new minimal Kac label s_2''. This algorithm is continued until all the Kac representations in (3.12) have been accounted for. This prescription also works for more complicated fusion products than $(1,s)\otimes(1,s')$ and always yields a unique and well-defined result, namely the fusion rules given in (3.7). Loosely speaking, the prescription corresponds to writing the decomposition of a fusion product in terms of Kac representations and then forming rank-2 modules whenever possible, starting with the lowest Kac label and moving up.

3.2.1 Full Kac fusion algebra

To describe the *full* Kac fusion algebra, not just its vertical component (3.6), we note that the horizontal component $\langle(r,1);\ r\in\mathbb{N}\rangle$ is characterized by the ordinary $sl(2)$ fusion rules

$$(r,1)\otimes(r',1) = \bigoplus_{r''=|r-r'|+1,\ \mathrm{by}\ 2}^{r+r'-1} (r'',1), \qquad r,r'\in\mathbb{N} \tag{3.13}$$

and that the lattice description implies not only (2.31) but also [7]

$$\mathcal{R}_r^b = (r,1) \otimes \mathcal{R}_1^b, \qquad r \in \mathbb{N} \tag{3.14}$$

The fusion rules of the full Kac fusion algebra now follow straightforwardly using the requirement of commutativity and associativity as we then have

$$\begin{aligned}
(r, b+kp) \otimes (r', b'+k'p) &= \big((r,1) \otimes (r',1)\big) \otimes \big((1, b+kp) \otimes (1, b'+k'p)\big) \\
\mathcal{R}_r^b \otimes (r', b'+k'p) &= \big((r,1) \otimes (r',1)\big) \otimes \big(\mathcal{R}_1^b \otimes (1, b'+k'p)\big) \\
\mathcal{R}_r^b \otimes \mathcal{R}_{r'}^{b'} &= \big((r,1) \otimes (r',1)\big) \otimes \big(\mathcal{R}_1^b \otimes \mathcal{R}_1^{b'}\big)
\end{aligned} \tag{3.15}$$

The last of these relations is not needed to determine the full Kac fusion algebra but must be satisfied for self-consistency of the fusion algebra. The fusion rules needed to complete the Kac fusion algebra are

$$\begin{aligned}
(r, b+kp) \otimes (r', b'+k'p) &= \bigoplus_{i=|r-r'|+1,\,\text{by}\,2}^{r+r'-1} \Big\{ \bigoplus_{j=|k-k'|+1,\,\text{by}\,2}^{k+k'-1} \ \bigoplus_{\ell=|i-j|+1,\,\text{by}\,2}^{i+j-1} \ \bigoplus_{\beta}^{p-|b-b'|-1} \mathcal{R}_\ell^\beta \\
&\oplus \bigoplus_{j=|k-k'+\text{sg}(b-b')|+1,\,\text{by}\,2}^{k+k'} \ \bigoplus_{\ell=|i-j|+1,\,\text{by}\,2}^{i+j-1} \ \bigoplus_{\beta}^{|b-b'|-1} \mathcal{R}_\ell^\beta \\
&\oplus \bigoplus_{\ell=|i-k-k'-1|+1,\,\text{by}\,2}^{i+k+k'} \ \bigoplus_{\beta}^{b+b'-p-1} \mathcal{R}_\ell^\beta \oplus \bigoplus_{\beta=|b-b'|+1,\,\text{by}\,2}^{p-|p-b-b'|-1} (i, \beta+(k+k')p) \Big\} \\
\mathcal{R}_r^b \otimes (r', b'+k'p) &= \Bigg(\Big\{ \bigoplus_{j=|r-k'-1|+1,\,\text{by}\,2}^{r+k'-2} \ \bigoplus_{\beta}^{p-|p-b-b'|-1} \oplus \bigoplus_{j=|r-k'-1+\text{sg}(p-b-b')|+1,\,\text{by}\,2}^{r+k'-\text{sg}(p-b-b')} \ \bigoplus_{\beta}^{|p-b-b'|-1} \\
&\oplus \bigoplus_{j=|r-k'|+1,\,\text{by}\,2}^{r+k'-1} \ \bigoplus_{\beta}^{p-|b-b'|-1} \oplus \bigoplus_{j=|r-k'+\text{sg}(b-b')|+1,\,\text{by}\,2}^{r+k'} \ \bigoplus_{\beta}^{|b-b'|-1} \Big\} \bigoplus_{\ell=|r'-j|+1,\,\text{by}\,2}^{r'+j-1} \mathcal{R}_\ell^\beta \\
&\oplus \bigoplus_{\ell=|r-r'+k'|+1,\,\text{by}\,2}^{r+r'+k'-1} \Big\{ \bigoplus_{\beta}^{b'-b-1} \oplus \bigoplus_{\beta=|b-b'|+1,\,\text{by}\,2}^{p-|p-b-b'|-1} \Big\} \mathcal{R}_\ell^\beta \Bigg) \Big/ (1+\delta_{b,0})
\end{aligned} \tag{3.16}$$

It follows, in particular, that the fundamental fusion algebra (3.1) is a subalgebra of the Kac fusion algebra (3.3). The fusion rules for critical dense polymers $\mathcal{LM}(1,2)$ are summarized in Appendix B.

Recalling that $\mathcal{R}_r^0 \equiv (1, rp) \equiv (r, p)$, it is noted that the modules

$$\big\{ \mathcal{R}_r^b;\ r \in \mathbb{N},\ b \in \mathbb{Z}_{0,p-1} \big\} \tag{3.17}$$

form an *ideal* of the Kac fusion algebra. This is in accordance with the expectation that these modules are *projective*.

3.2.2 Evidence for the fusion conjecture: lattice approach

The lattice approach to the logarithmic minimal model $\mathcal{LM}(1,p)$ [1, 7] is based on a loop model with loop fugacity

$$\beta = -2\cos\tfrac{\pi}{p} \tag{3.18}$$

Here we are interested in the model defined on strips of width N. To describe the vertical Kac representations and their fusions, it suffices to consider the hamiltonian defined by

$$H = -\sum_{j=1}^{N-1} e_j \tag{3.19}$$

where $\{e_j;\ j \in \mathbb{Z}_{1,N-1}\}$ is the set of Temperley-Lieb generators acting on N strands. Empirically [1], the character of the Kac representation $(1,s)$ arises in the scaling limit of the spectrum of the hamiltonian acting on link states with exactly $s-1$ defects. Viewing these defects as linked to the right (or left) boundary, the Kac representation is associated with the corresponding boundary condition. In particular, there are $N-1$ link states with exactly $N-2$ defects and our choice of canonical ordering of these link states is

$$\bigcap\,|\,|\ldots|\,|,\qquad |\bigcap|\ldots|\,|,\qquad \ldots\ldots,\qquad |\,|\ldots|\,|\bigcap \tag{3.20}$$

We refer to [28] for more details.

Fusion is implemented diagrammatically by considering non-trivial boundary conditions on *both* sides of the bulk. In the diagrammatic description of the fusion product $(1,s)\otimes(1,s')$, there are thus $s-1$ and $s'-1$ links emanating from the left and right boundaries, respectively. As links from the left boundary can be joined with links from the right boundary to form half-arcs above the bulk, the number of defects propagating through the bulk is given by $s+s'-2-2t$ where $0 \le t \le \min\{s,s'\}-1$. In the last expression in (3.9), the integer t labels the number of such half-arcs linking the two boundaries. For given t, we thus have $s-t-1$ and $s'-t-1$ half-arcs linking the bulk to the left and right boundary, respectively.

As usual, we group the link states according to their number of half-arcs linking the bulk to the boundaries and order these groups with increasing such numbers. The resulting matrix representation of the hamiltonian is then upper block-triangular with vanishing blocks beyond the first super-diagonal of blocks. It is recalled that we do not anticipate Jordan cells of ranks greater than 2 in the hamiltonian. To examine Jordan cells of rank 2 formed between *neighbouring* blocks on the diagonal, it thus suffices to analyze the upper block-triangular matrix defined by the four adjacent blocks spanned diagonally by the said two blocks. This gives insight into the appearance of rank-2 modules of the type $\mathcal{R}_r^1$. Beyond neighbouring blocks, care has to be taken, though, since some non-trivial Jordan cells are formed using 'ligatures', see (3.21) below. This is indeed the case for $\mathcal{R}_r^b$ for $b>1$ since such a rank-2 module can be viewed as an indecomposable sum (3.10) of two Kac representations corresponding to boundary conditions differing in numbers of defects by $2b>2$. The responsible Jordan cells are thus formed between blocks which are *not* neighbours.

As illustration of this 'ligature phenomenon', we consider the matrix

$$M = \begin{pmatrix} a & 1 & 0 \\ 0 & b & 1 \\ 0 & 0 & a \end{pmatrix} \tag{3.21}$$

For $a \neq b$, its Jordan canonical form reads

$$J = S^{-1}MS = \begin{pmatrix} b & 0 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix} \tag{3.22}$$

where

$$S^{-1} = \begin{pmatrix} 0 & -\sigma & 1 \\ \sigma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\qquad S = \begin{pmatrix} \sigma^{-2} & \sigma^{-1} & -\sigma^{-2} \\ -\sigma^{-1} & 0 & \sigma^{-1} \\ 0 & 0 & 1 \end{pmatrix},\qquad \sigma = a-b \tag{3.23}$$

That is, a rank-2 Jordan cell is formed between the two copies of the degenerate eigenvalue a. If, on the other hand, we eliminate the second row and column from M *before* examining the possibility of a rank-2 Jordan cell, we end up with the *diagonal* matrix $\mathrm{diag}(a,a)$. A search for non-trivial Jordan cells can therefore not be conducted this naively, and focus here is on neighbouring blocks. That is, we are only concerned with the appearance of rank-2 modules of the type $\mathcal{R}_r^1$. It is also noted that permutations alone cannot resolve the indicated problems associated with treating blocks which are not neighbours. This is again illustrated by the matrix M in (3.21) which is similar to

$$P^{-1}MP = \begin{pmatrix} a & 0 & 1 \\ 0 & a & 0 \\ 0 & 1 & b \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.24}$$

However, the matrix $P^{-1}MP$ is not upper block-triangular with vanishing blocks beyond the first super-diagonal of blocks.

Now, let us implement the fusion product $(1,s)\otimes(1,s')$ for $s,s'>1$ on a lattice of limited system size

$$N = s+s'-2-2t, \qquad t = 0,1,\ldots,\min\{s,s'\}-1 \tag{3.25}$$

for some t in the range given. This means that the bulk can accommodate up to N defects while there must be at least t half-arcs linking the two boundaries. In the decomposition (3.8), the t rightmost Kac representations are therefore not present while the remaining ones are

$$(1,s)\otimes(1,s')\big|_N = (1,|s-s'|+1) \overset{?}{\oplus} \ldots \overset{?}{\oplus} (1,s+s'-3-2t) \overset{?}{\oplus} (1,s+s'-1-2t) \tag{3.26}$$

To gain insight into whether the final sum in this decomposition is *direct* or *indecomposable*, we will now characterize when a non-trivial Jordan cell is formed in the hamiltonian $H_{s,s'}^{(N)}$ between the two neighbouring blocks corresponding to $N-2$ or N defects, respectively. For $N=6$, using the ordered basis (3.20), the corresponding matrix realization of the hamiltonian is given by

$$-H_{s,s'}^{(6)} = \begin{pmatrix} \beta & 1 & 0 & 0 & 0 & \delta_{s-t,2} \\ 1 & \beta & 1 & 0 & 0 & \delta_{s-t,3} \\ 0 & 1 & \beta & 1 & 0 & \delta_{s-t,4} \\ 0 & 0 & 1 & \beta & 1 & \delta_{s-t,5} \\ 0 & 0 & 0 & 1 & \beta & \delta_{s-t,6} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.27}$$

The extension to general N is straightforward and discussed in Appendix C. We thus find that $H_{s,s'}^{(N)}$ is diagonalizable unless there exists $j_0\in\mathbb{Z}_{1,N-1}$ for which $\beta+2\cos\frac{j_0\pi}{N}=0$ and $\sin\frac{j_0(s-t-1)\pi}{N}\neq 0$ in which case the Jordan canonical form of $H_{s,s'}^{(N)}$ contains a single non-trivial Jordan cell. This cell is of rank 2 and has diagonal elements 0. It follows that this non-trivial Jordan cell appears if and only if

$$p\mid(s+s'-2-2t), \qquad p\nmid(s-t-1) \tag{3.28}$$

Since the pair of conditions $q\,|\,(n+m)$ and $q\nmid n$ implies $q\nmid m$, we may restore the symmetry between s and s' in (3.28) by redundantly including $p\nmid(s'-t-1)$. This symmetry is a manifestation of the commutativity of the fusion product $(1,s)\otimes(1,s')$, of the equivalence of the left- and right-sided decompositions of the diagrammatic implementation of this fusion product, and of the choice of canonical ordering of the link states with $N-2$ defects (3.20).

This *exact* result (3.28) for finite system sizes is in accordance with the fusion rule (3.7) for $(1,s)\otimes(1,s')$. Indeed, assuming that the observed Jordan-cell structures survive in the continuum scaling limit, the result provides valuable insight used to determine whether the particular sum

$$(1,s)\otimes(1,s') = \ldots(1,s+s'-3-2t) \overset{?}{\oplus} (1,s+s'-1-2t)\ldots \tag{3.29}$$

in the decomposition (3.8) of the fusion product is *direct* or *indecomposable*. From the lattice analysis above, we thus conclude that it is indecomposable due to the presence of non-trivial Jordan cells if and only if the conditions in (3.28) are satisfied. For this to be compatible with the conjectured fusion rules, the latter must predict that the rank-2 module $\mathcal{R}_r^1$ appears (with multiplicity 1) in the decomposition of $(1,s)\otimes(1,s')$ if and only if

$$\exists\,\tau\in\mathbb{Z}_{1,\min\{s,s'\}-1}: \qquad rp = |s-s'|+2\tau, \qquad \tau\notin p\mathbb{N} \tag{3.30}$$

Writing $\tau = a+\ell p$, this is easily verified.

From the lattice approach, we now know where certain Jordan cells appear in the decomposition of $(1,s)\otimes(1,s')$, but in general, this is not sufficient to determine the various representations. In critical dense polymers $\mathcal{LM}(1,2)$, for example, we have thus found that

$$(1,3)\otimes(1,3) = (1,1)\oplus_i(1,3)\overset{?}{\oplus}(1,5) \tag{3.31}$$

where the indecomposable sum is due to the formation of non-trivial Jordan cells. The lattice approach offers an additional clue. Continuing the examination of (3.31), we note that the link states associated with the subfactor $(1,1)$ (corresponding to $t=2$) and the link states associated with the subfactor $(1,3)$ (corresponding to $t=1$) all contain a half-arc linking the two boundaries. Ignoring this common spectator half-arc, the diagrammatic description becomes equivalent to the lattice implementation of the fusion product

$$(1,2)\otimes(1,2) = \mathcal{R}_1^1 \tag{3.32}$$

Alternatively, we may focus on the link states associated with the subfactors $(1,3)$ and $(1,5)$ corresponding to $t=1$ or $t=0$, respectively. Unlike before, this does not correspond to a single fusion product. The only candidate with the same number of defects propagating through the bulk is $(1,2)\otimes(1,4)$, but this is associated with link states with 1 and 3 links emanating from the left and right boundaries, respectively. We thus conclude that the fusion product $(1,3)\otimes(1,3)$ contains the rank-2 module $\mathcal{R}_1^1$ as a subfactor, that is,

$$(1,3)\otimes(1,3) = \mathcal{R}_1^1\overset{?}{\oplus}(1,5) \tag{3.33}$$

Below, we supplement this lattice analysis of the fusion product $(1,3)\otimes(1,3)$ by applications of the NGK algorithm.

3.2.3 Evidence for the fusion conjecture: NGK algorithm

A priori, the right side of (3.33) could correspond to a single indecomposable representation (since it remains to be established that (3.17) is the set of projective representations). According to the conjectured fusion rules (3.7), however, the full decomposition reads

$$(1,3)\otimes(1,3) = \mathcal{R}_1^1\oplus(1,5) \tag{3.34}$$

To test this, we have applied the NGK algorithm to the fusion product $(1,3)\otimes(1,3)$, assuming that $(1,3)$ is a highest-weight module. Details of this analysis to Nahm level 2 appear in Appendix A.5,

and they confirm the fusion rule (3.34). They also confirm that the Kac representation $(1,5)$ is a highest-weight module and not its contragredient module. Likewise in $\mathcal{LM}(1,2)$, we have confirmed the fusion rule

$$(1,3)\otimes(1,5)=\mathcal{R}_2^1\oplus(1,7) \tag{3.35}$$

and the highest-weight property of $(1,7)$ to Nahm level 3.

As observed in Section 3.4 below, the vertical Kac representations $(1,s)$ are all generated from repeated fusion of $(1,2)$ and $(1,p+1)$. In accordance with the results of the NGK algorithm, it is therefore natural to expect that the Kac representations $(1,s)$ thereby generated are all highest-weight modules provided that $(1,p+1)$ is. It thus *suffices* to assume that $(1,p+1)$ is a highest-weight module.

Refined highest-weight assumption. (i) The Kac representation $(1,p+1)$ is a highest-weight Virasoro module. (ii) Repeated fusion subsequently ensures that all vertical Kac representations $(1,s)$ are highest-weight Virasoro modules.

Our analysis does not, however, provide direct arguments for the assumption that the Kac representation $(1,p+1)$ is a highest-weight module. As we will see below, the fusion rules actually turn out to be independent of whether $(1,p+1)$ is indeed a highest-weight module or the corresponding contragredient module.

3.2.4 Even and odd sectors

From the lattice approach, it is of interest to understand the continuum scaling limit of the situation where the only constraint on the number of defects is that it is of the same parity as the bulk system size N. Depending on the parity of N, we refer to the two possible scenarios as the *even* and *odd sectors*. They can be viewed as systems with *free boundary conditions*, but they can also be interpreted as finitized versions of the fusion products

$$(1,\tfrac{N+2}{2})\otimes(1,\tfrac{N+2}{2})\quad\text{and}\quad(1,\tfrac{N+1}{2})\otimes(1,\tfrac{N+3}{2}) \tag{3.36}$$

respectively. To examine the continuum scaling limit of a system with free boundary conditions, we can thus resort to the fusion rules for the fusions in (3.36) as given in (3.7). For $b\in\mathbb{Z}_{0,p-1}$ and $k\in\mathbb{N}_0$, the first fusion rule in (3.7) yields

$$\begin{aligned}(1,b+kp)\otimes(1,b+kp) &= \bigoplus_{j=1}^{k}\bigoplus_{\beta}^{p-1}\mathcal{R}_{2j-1}^{\beta}\oplus\bigoplus_{\beta}^{2b-p-1}\mathcal{R}_{2k+1}^{\beta}\oplus\bigoplus_{\beta}^{p-|p-2b|-1}(1,\beta+2kp)\\ (1,b+kp)\otimes(1,b+1+kp) &= \bigoplus_{j=1}^{k}\bigoplus_{\beta}^{p-2}\mathcal{R}_{2j-1}^{\beta}\oplus\bigoplus_{\beta}^{2b-p}\mathcal{R}_{2k+1}^{\beta}\oplus\bigoplus_{j=1}^{k}\mathcal{R}_{2j}^{0}\oplus\bigoplus_{\beta=2,\,\text{by}\,2}^{p-|p-2b-1|-1}(1,\beta+2kp)\end{aligned} \tag{3.37}$$

It is verified that the second of these rules applies for $b=p-1$, even though $b'=p$ in that case. It follows that the continuum scaling limit of a system with free boundary conditions is described by

$$\lim_{n\to\infty}(1,n)\otimes(1,n)=\bigoplus_{j\in\mathbb{N}}\bigoplus_{\beta}^{p-1}\mathcal{R}_{2j-1}^{\beta},\qquad \lim_{n\to\infty}(1,n)\otimes(1,n+1)=\bigoplus_{j\in\mathbb{N}}\Big(\mathcal{R}_{2j}^{0}\oplus\bigoplus_{\beta}^{p-2}\mathcal{R}_{2j-1}^{\beta}\Big) \tag{3.38}$$

in accordance with the recent analysis of Jordan structures in [29]. In particular, for critical dense polymers as described by $\mathcal{LM}(1,2)$ [28], we thus have

$$\lim_{n\to\infty}(1,n)\otimes(1,n)=\bigoplus_{j\in\mathbb{N}}\mathcal{R}_{2j-1}^{1},\qquad \lim_{n\to\infty}(1,n)\otimes(1,n+1)=\bigoplus_{j\in\mathbb{N}}(1,2j) \tag{3.39}$$

showing that reducible yet indecomposable representations only arise in the even sector.

3.3 Contragredient extension

It is stressed that the set

$$\mathcal{J}^{\text{Kac}} = \big\{(r,s), \mathcal{R}_r^b;\ r,s \in \mathbb{N},\ b \in \mathbb{Z}_{1,p-1}\big\} \tag{3.40}$$

of representations appearing in the Kac fusion algebra exhausts the set of representations associated with boundary conditions in [1, 7]. Extending this set by the contragredient Kac representations

$$\mathcal{J}^{\text{Kac}} \to \mathcal{J}^{\text{Cont}} = \mathcal{J}^{\text{Kac}} \cup \big\{(r,s)^*;\ r,s \in \mathbb{N}\big\} \tag{3.41}$$

gives rise to the larger fusion algebra

$$\big\langle \mathcal{J}^{\text{Cont}} \big\rangle = \big\langle (r,s), (r,s)^*, \mathcal{R}_r^b;\ r,s \in \mathbb{N},\ b \in \mathbb{Z}_{1,p-1} \big\rangle \tag{3.42}$$

where we recall (2.40). A priori, additional representations could be generated by repeated fusion of the representations listed. However, preliminary evaluations of a variety of fusion products seem to suggest that the extended fusion algebra (3.42) closes on the set of representations listed. To describe this fusion algebra, we introduce

$$\mathcal{C}_n[(r,s)] = \begin{cases} (r,s), & n>0 \\ (r,s)^*, & n<0 \end{cases} \tag{3.43}$$

In our applications, $\mathcal{C}_0[(r,s)]$ only appears if (r,s) is irreducible in which case

$$\mathcal{C}_0[(r,s)] = (r,s) = (r,s)^*, \qquad s \in \mathbb{Z}_{1,p-1} \cup p\mathbb{N} \tag{3.44}$$

where we have extended the definition of $\mathcal{C}_0$ to all fully reducible representations (2.40).

Contragredient fusion conjecture. The fusion rules involving contragredient Kac representations in the extended fusion algebra (3.42) are given by or follow readily from

$$(r,s)^* \otimes (r',s')^* = \big((r,s)\otimes(r',s')\big)^*, \qquad \mathcal{R}_r^b \otimes (r',s')^* = \mathcal{R}_r^b \otimes (r',s') \tag{3.45}$$

and

$$\begin{aligned}(1,b+kp)\otimes(1,b'+k'p)^* \;=\; & \bigoplus_{j=|k-k'|+2,\,\text{by}\,2}^{k+k'} \ \bigoplus_{\beta}^{p-|p-b-b'|-1} \mathcal{R}_j^\beta \oplus \bigoplus_{j=|k-k'|+1,\,\text{by}\,2}^{k+k'-\text{sg}(p-b-b')} \ \bigoplus_{\beta}^{|p-b-b'|-1} \mathcal{R}_j^\beta \\ & \oplus \bigoplus_{\beta}^{(b-b')\text{sg}(k'-k)-1} \mathcal{R}_{|k-k'|}^\beta \oplus \bigoplus_{\beta=|b-b'|+1,\,\text{by}\,2}^{p-|p-b-b'|-1} \mathcal{C}_{k-k'}[(1,\beta+|k-k'|p)]\end{aligned} \tag{3.46}$$

where $b,b' \in \mathbb{Z}_{0,p-1}$ and $k,k' \in \mathbb{N}_0$.

Since $(r,1)$ is irreducible, we thus have

$$(r,s)^* = (r,1)^* \otimes (1,s)^* = (r,1)\otimes(1,s)^* \tag{3.47}$$

from which it follows that the general fusion product $(r,s)\otimes(r',s')^*$ can be computed as

$$(r,s)\otimes(r',s')^* = \big((r,1)\otimes(r',1)\big)\otimes\big((1,s)\otimes(1,s')^*\big) \tag{3.48}$$

This yields the general fusion rule

$$\begin{aligned}
(r,b+kp)\otimes(r',b'+k'p)^* \;\; &= \bigoplus_{i=|r-r'|+1,\,\mathrm{by}\,2}^{r+r'-1}\Big\{\bigoplus_{j=|k-k'|+2,\,\mathrm{by}\,2}^{k+k'}\;\bigoplus_{\ell=|i-j|+1,\,\mathrm{by}\,2}^{i+j-1}\;\bigoplus_{\beta}^{p-|p-b-b'|-1}\mathcal{R}_\ell^\beta\\
&\oplus\bigoplus_{j=|k-k'|+1,\,\mathrm{by}\,2}^{k+k'-\mathrm{sg}(p-b-b')}\;\bigoplus_{\ell=|i-j|+1,\,\mathrm{by}\,2}^{i+j-1}\;\bigoplus_{\beta}^{|p-b-b'|-1}\mathcal{R}_\ell^\beta\\
&\oplus\bigoplus_{\ell=|i-|k-k'||+1,\,\mathrm{by}\,2}^{i+|k-k'|-1}\;\bigoplus_{\beta}^{(b-b')\mathrm{sg}(k'-k)-1}\mathcal{R}_\ell^\beta\\
&\oplus\bigoplus_{\beta=|b-b'|+1,\,\mathrm{by}\,2}^{p-|p-b-b'|-1}\mathcal{C}_{k-k'}[(i,\beta+|k-k'|p)]\Big\}
\end{aligned}\tag{3.49}$$

In general, the fusion rules are not invariant under replacement by contragredient modules as illustrated by

$$(1,1)^*\otimes(r,s)=(r,s)\neq(r,s)^*=(1,1)\otimes(r,s)^*,\qquad p<s\neq kp \tag{3.50}$$

These trivial fusion rules are encoded in (3.49) and correspond to $r'=1,b'=1,k'=0$ or $r=1,b=1,k=0$, respectively. As a consequence of (3.45), we note that the extended fusion algebra contains the two isomorphic fusion subalgebras

$$\big\langle(r,s),\mathcal{R}_r^b;\,r,s\in\mathbb{N},\,b\in\mathbb{Z}_{1,p-1}\big\rangle\simeq\big\langle(r,s)^*,\mathcal{R}_r^b;\,r,s\in\mathbb{N},\,b\in\mathbb{Z}_{1,p-1}\big\rangle \tag{3.51}$$

of which the first one is the Kac fusion algebra. It is also noted that the representations in (3.17) form an *ideal* of the extended fusion algebra, still in accordance with the representations being projective.

The fusion rules (3.46) can be obtained by extending the applications of the forgetful functor (3.11) with

$$\mathcal{F}[(1,s)^*]=(1,s) \tag{3.52}$$

and subsequently modifying the prescription or algorithm discussed following (3.12). In that discussion, we formed rank-2 modules starting with the *lowest* Kac label – now we start with the *greatest* Kac label. That is, after applying the forgetful functor to the fusion product $(1,s)\otimes(1,s')^*$

$$\mathcal{F}[(1,s)\otimes(1,s')^*]=\bigoplus_{s''=|s-s'|+1,\,\mathrm{by}\,2}^{s+s'-1}(1,s'') \tag{3.53}$$

we initially focus on the Kac representation $(1,s''_1)$ with maximal Kac label $s''_1=s+s'-1$. Depending on p, this will appear as the submodule $(1,rp+b)$ of the rank-2 module $\mathcal{R}_r^b$ if and only if the matching module $(1,rp-b)$ also appears in the decomposition in (3.53). If not, the (contragredient) Kac representation $\mathcal{C}_{s-s'}[(1,s''_1)]$ will appear 'by itself' in the decomposition of the fusion product. If a rank-2 module is not formed for $s=s'$, the two options $(1,s''_1)$ and $(1,s''_1)^*$ turn out to be identical. Having completed the examination of $(1,s''_1)$, we remove it together with its potential partner $(1,rp-b)$ from the direct sum in (3.53) and repeat the analysis for $(1,s''_2)$ corresponding to the new maximal Kac label s''_2. As before, this algorithm is continued until all the Kac representations in (3.53) have been accounted for. It is straightforward to verify that this prescription yields the fusion rules (3.46).

3.4 Polynomial fusion rings

Together with the fact that the fundamental fusion algebra is a subalgebra of the Kac fusion algebra, the fusion rules

$$\begin{aligned}(1,2)\otimes(1,kp+b) &= (1,kp+b-1)\oplus(1,kp+b+1)\\ (1,p+1)\otimes(1,kp+b) &= \bigoplus_{\beta}^{p-b}\mathcal{R}_k^\beta\oplus\bigoplus_{\beta}^{b-2}\mathcal{R}_{k+1}^\beta\oplus(1,(k+1)p+b)\end{aligned}\tag{3.54}$$

demonstrate that the Kac fusion algebra is generated from repeated fusion of the Kac representations

$$\big\{(1,1),(2,1),(1,2),(1,p+1)\big\}\tag{3.55}$$

that is,

$$\big\langle\mathcal{J}^{\mathrm{Kac}}\big\rangle=\big\langle(r,s);\ r,s\in\mathbb{N}\big\rangle=\big\langle(1,1),(2,1),(1,2),(1,p+1)\big\rangle\tag{3.56}$$

It is therefore natural to expect that this fusion algebra is isomorphic to a polynomial ring in the three entities $X\leftrightarrow(2,1)$, $Y\leftrightarrow(1,2)$ and $Z\leftrightarrow(1,p+1)$. This is indeed what we find.

Proposition 1. The Kac fusion algebra is isomorphic to the polynomial ring generated by X, Y and Z modulo the ideal $(P_p(X,Y),Q_p(Y,Z))$, that is,

$$\big\langle\mathcal{J}^{\mathrm{Kac}}\big\rangle\simeq\mathbb{C}[X,Y,Z]/\big(P_p(X,Y),Q_p(Y,Z)\big)\tag{3.57}$$

where

$$P_p(X,Y)=\big[X-2T_p(\tfrac{Y}{2})\big]U_{p-1}(\tfrac{Y}{2}),\qquad Q_p(Y,Z)=\big[Z-U_p(\tfrac{Y}{2})\big]U_{p-1}(\tfrac{Y}{2})\tag{3.58}$$

For $r\in\mathbb{N}$, $k\in\mathbb{N}_0$ and $b\in\mathbb{Z}_{0,p-1}$, the isomorphism reads

$$\begin{aligned}(r,kp+b) &\leftrightarrow U_{r-1}(\tfrac{X}{2})\Big(U_{kp+b-1}(\tfrac{Y}{2})+\big[Z^k-U_p^k(\tfrac{Y}{2})\big]U_{b-1}(\tfrac{Y}{2})\Big)\\ \mathcal{R}_r^b &\leftrightarrow (2-\delta_{b,0})U_{r-1}(\tfrac{X}{2})T_b(\tfrac{Y}{2})U_{p-1}(\tfrac{Y}{2})\end{aligned}\tag{3.59}$$

where $T_n(x)$ and $U_n(x)$ are Chebyshev polynomials of the first and second kind, respectively.

Proof. The relation $P_p(X,Y)=0$ corresponds to the identification $(2,p)\equiv(1,2p)$ and encodes $(r,p)\equiv(1,rp)$ more generally, cf. (3.63), while the relation $Q_p(Y,Z)=0$ follows from the fusion rule

$$(1,p)\otimes(1,p+1)=\bigoplus_{\beta}^{p-2}\mathcal{R}_1^\beta\oplus(1,2p)\tag{3.60}$$

The remaining fusion rules are then verified straightforwardly in the polynomial ring. Here we only demonstrate the two fusion rules in (3.54). The first of these follows immediately from the recursion relation for the Chebyshev polynomials. To show the second of the fusion rules, we note the basic decomposition rules

$$U_m(x)U_n(x)=\sum_{j=|m-n|,\,\mathrm{by}\,2}^{m+n}U_j(x),\qquad 2T_m(x)U_{n-1}(x)=U_{n+m-1}(x)+\mathrm{sg}(n-m)U_{|n-m|-1}(x)\tag{3.61}$$

where $U_{-1}(x)=0$. As a consequence, we have

$$U_{p-1}(x)\sum_{j=0}^{k-1}U_p^{k-j-1}(x)U_{jp+b-2}(x)=U_{b-1}(x)U_p^k(x)-U_{kp+b-1}(x)\tag{3.62}$$

which is established by induction in k and shows that the expression on the right side is divisible by $U_{p-1}(x)$. This is of importance when multiplied by Z due to the form of $Q_p(Y,Z)$. With the additional observation that

$$U_{r-1}(\tfrac{X}{2})U_{p-1}(\tfrac{Y}{2}) \equiv U_{rp-1}(\tfrac{Y}{2}) \pmod{P_p(X,Y)} \tag{3.63}$$

which follows by induction in r, the second fusion rule readily follows. $\square$

Extending the arguments just presented for the Kac fusion algebra, one finds that the extended Kac fusion algebra (3.42) is also generated from repeated fusion of a small number of Kac representations

$$\big\langle (r,s),(r,s)^*;\ r,s\in\mathbb{N}\big\rangle = \big\langle (1,1),(2,1),(1,2),(1,p+1),(1,p+1)^*\big\rangle \tag{3.64}$$

and that it is isomorphic to a polynomial ring.

Proposition 2. The extended Kac fusion algebra (3.42) is isomorphic to the polynomial ring generated by X, Y, Z and Z^* modulo the ideal $(P_p(X,Y),Q_p(Y,Z),Q_p(Y,Z^*),R_p(Y,Z,Z^*))$, that is,

$$\big\langle \mathcal{J}^{\mathrm{Cont}}\big\rangle \simeq \mathbb{C}[X,Y,Z,Z^*]/\big(P_p(X,Y),Q_p(Y,Z),Q_p(Y,Z^*),R_p(Y,Z,Z^*)\big) \tag{3.65}$$

where the polynomials P_p and Q_p are defined in (3.58) while

$$R_p(Y,Z,Z^*) = ZZ^* - U_p^2(\tfrac{Y}{2}) \tag{3.66}$$

For $r\in\mathbb{N}$, $k\in\mathbb{N}_0$ and $b\in\mathbb{Z}_{0,p-1}$, the isomorphism reads

$$\begin{aligned}
(r,kp+b) \quad &\leftrightarrow \quad U_{r-1}(\tfrac{X}{2})\Big(U_{kp+b-1}(\tfrac{Y}{2}) + \big[Z^k - U_p^k(\tfrac{Y}{2})\big]U_{b-1}(\tfrac{Y}{2})\Big)\\
(r,kp+b)^* \quad &\leftrightarrow \quad U_{r-1}(\tfrac{X}{2})\Big(U_{kp+b-1}(\tfrac{Y}{2}) + \big[(Z^*)^k - U_p^k(\tfrac{Y}{2})\big]U_{b-1}(\tfrac{Y}{2})\Big)\\
\mathcal{R}_r^b \quad &\leftrightarrow \quad (2-\delta_{b,0})U_{r-1}(\tfrac{X}{2})T_b(\tfrac{Y}{2})U_{p-1}(\tfrac{Y}{2})
\end{aligned} \tag{3.67}$$

Proof. Compared to the proof of Proposition 1, the essential new feature is the appearance of Z^*. The relation $Q_p(Y,Z^*)=0$ plays the same role for the contragredient Kac representations and Z^* as $Q_p(Y,Z)=0$ does for the Kac representations and Z. This yields the part of the polynomial ring corresponding to (3.51). The relation $R_p(Y,Z,Z^*)=0$ corresponds to the fusion rule

$$(1,p+1)\otimes(1,p+1)^* = (1,1)\oplus\bigoplus_{\beta}^{p-3}\mathcal{R}_1^\beta\oplus\mathcal{R}_2^1 \tag{3.68}$$

To establish the general fusion rule (3.46) in the ring picture, we first use induction in n to establish

$$U_p^{2n}(\tfrac{Y}{2})Z^m \equiv Z^m + \sum_{j=0}^{n-1}U_p^{m+2j}(\tfrac{Y}{2})U_{p-1}(\tfrac{Y}{2})U_{p+1}(\tfrac{Y}{2}) \pmod{Q_p(Y,Z)}, \qquad n\in\mathbb{N} \tag{3.69}$$

and similarly for Z replaced by Z^*. This is needed when reducing

$$Z^k(Z^*)^{k'} \equiv U_p^{2\min(k,k')}(\tfrac{Y}{2})\begin{cases} Z^{k-k'}, & k\geq k'\\ (Z^*)^{k'-k}, & k<k'\end{cases} \pmod{R_p(Y,Z,Z^*)} \tag{3.70}$$

For simplicity, we let $k\geq k'$ in which case we find

$$(1,b+kp)\otimes(1,b'+k'p)^* \leftrightarrow \big[Z^{k-k'} - U_p^{k-k'}(\tfrac{Y}{2})\big]U_{b-1}(\tfrac{Y}{2})U_{b'-1}(\tfrac{Y}{2}) + U_{kp+b-1}(\tfrac{Y}{2})U_{k'p+b'-1}(\tfrac{Y}{2}) \tag{3.71}$$

This polynomial expression is recognized as corresponding to the right side of (3.46). $\square$

4 Conclusion

We have discussed the representation content and fusion algebras of the logarithmic minimal model $\mathcal{LM}(1,p)$. We have thus proposed a classification of the entire family of Kac representations as submodules of Feigin-Fuchs modules and presented a conjecture for their fusion algebra. To test these proposals, we have used a combination of the lattice approach to $\mathcal{LM}(1,p)$ and applications of the NGK algorithm. We have also discussed a natural extension of the representation content by inclusion of the modules contragredient to the Kac representations, and we have presented a conjecture for the corresponding fusion algebra. This extended fusion algebra as well as the conjecture for the Kac fusion algebra itself were then shown to be isomorphic to polynomial fusion rings which were described explicitly.

Continuing the work in [30] on a Kazhdan-Lusztig-dual quantum group for the logarithmic minimal model $\mathcal{LM}(1,p)$, fusion of Kac representations is considered in [31]. The corresponding fusion algebra appears to be equivalent to the one discussed here. This is a very reassuring observation for both methodologies and offers independent evidence for the Kac fusion algebra discussed here.

The work presented here pertains to the logarithmic minimal models $\mathcal{LM}(1,p)$, but the methods used in obtaining the various results are expected to extend straightforwardly to the general family of logarithmic minimal models $\mathcal{LM}(p,p')$. We hope to discuss the corresponding classification of Kac representations and their fusion algebras elsewhere. The case $\mathcal{LM}(2,3)$ is particularly interesting as it describes critical percolation.

We find the remarkably simple expression (2.32) in the singular vector conjecture very intriguing. Preliminary results indicate that it can be extended from $\Delta_{2,1}$ to general $\Delta_{r,1}$ and even to general logarithmic minimal models $\mathcal{LM}(p,p')$. We also hope to discuss this elsewhere.

In the $\mathcal{W}$-extended picture $\mathcal{WLM}(1,p)$, Yang-Baxter integrable boundary conditions associated with irreducible or projective representations of the triplet $\mathcal{W}$-algebra $\mathcal{W}(p)$ were introduced in [15]. With the results of the present work, it is natural to expect that there also exist Yang-Baxter integrable boundary conditions associated with the reducible yet indecomposable $\mathcal{W}(p)$-representations of rank 1 appearing in [32]. This is indeed what we find as we will discuss elsewhere [33].

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A Nahm-Gaberdiel-Kausch algorithm

Here we summarize some of the ingredients in the NGK algorithm, but refer to the original papers [8, 9] as well as [10, 11] for more details.

A.1 Co-multiplication

We are interested in the co-multiplications given by

$$\Delta(L_n) = \sum_{m=-1}^{n} \binom{n+1}{m+1} L_m \times I + I \times L_n, \qquad n \in \mathbb{Z}_{\geq -1} \tag{A.1}$$

and

$$
\begin{aligned}
\Delta(L_{-n}) &= \sum_{m=-1}^{\infty} (-1)^{m+1} \binom{n+m-1}{m+1} L_m \times I + I \times L_{-n}, && n \in \mathbb{Z}_{\geq 2} \\
\tilde{\Delta}(L_{-n}) &= L_{-n} \times I + (-1)^{n+1} \sum_{m=-1}^{\infty} \binom{n+m-1}{n-2} I \times L_m, && n \in \mathbb{Z}_{\geq 2}
\end{aligned}
\tag{A.2}
$$

This should not be confused with the notation for conformal weights. Useful examples of the co-multiplications are

$$
\begin{aligned}
\Delta(L_2) &= L_{-1} \times I + 3L_0 \times I + 3L_1 \times I + L_2 \times I + I \times L_2 \\
\Delta(L_1) &= L_{-1} \times I + 2L_0 \times I + L_1 \times I + I \times L_1 \\
\Delta(L_0) &= L_{-1} \times I + L_0 \times I + I \times L_0 \\
\Delta(L_{-1}) &= L_{-1} \times I + I \times L_{-1} \\
\Delta(L_{-2}) &= \big(\ldots - L_2 \times I + L_1 \times I - L_0 \times I + L_{-1} \times I\big) + I \times L_{-2} \\
\tilde{\Delta}(L_{-2}) &= L_{-2} \times I - \big(\ldots + I \times L_2 + I \times L_1 + I \times L_0 + I \times L_{-1}\big)
\end{aligned}
\tag{A.3}
$$

A.2 Singular vectors

We denote the singular vector appearing at level rs in the highest-weight Verma module $V_{r,s}$ by $|\lambda_{r,s}\rangle$ and normalize it by setting the coefficient to L_{-1}^{rs} equal to 1. For $r = 1, 2, 3, 4, 5$, the singular vector $|\lambda_{r,1}\rangle$ is given by

$$
\begin{aligned}
|\lambda_{1,1}\rangle &= \big\{L_{-1}\big\}|\Delta_{1,1}\rangle \\
|\lambda_{2,1}\rangle &= \big\{L_{-1}^2 - pL_{-2}\big\}|\Delta_{2,1}\rangle \\
|\lambda_{3,1}\rangle &= \big\{L_{-1}^3 - 4pL_{-2}L_{-1} + 2p(2p-1)L_{-3}\big\}|\Delta_{3,1}\rangle \\
|\lambda_{4,1}\rangle &= \big\{L_{-1}^4 - 10pL_{-2}L_{-1}^2 + 9p^2L_{-2}^2 + 2p(12p-5)L_{-3}L_{-1} - 6p(6p^2-4p+1)L_{-4}\big\}|\Delta_{4,1}\rangle \\
|\lambda_{5,1}\rangle &= \big\{L_{-1}^5 - 20pL_{-2}L_{-1}^3 + 64p^2L_{-2}^2L_{-1} + 6p(14p-5)L_{-3}L_{-1}^2 - 64p^2(3p-1)L_{-3}L_{-2} \\
&\quad - 12p(24p^2-14p+3)L_{-4}L_{-1} + 8p(3p-1)(24p^2-14p+3)L_{-5}\big\}|\Delta_{5,1}\rangle
\end{aligned}
\tag{A.4}
$$

The singular vectors $|\lambda_{1,s}\rangle$ follow from these by application of the general relation

$$
|\lambda_{r,s}\rangle = |\lambda_{s,r}\rangle\big|_{p\to 1/p} \tag{A.5}
$$

A.3 Nahm level, basis and spurious subspace

The enveloping algebra of the Virasoro algebra contains the subalgebra

$$
\mathcal{A}_n = \big\langle \prod_{j=1}^{m} L_{-\ell_j};\ \sum_{j=1}^{m} \ell_j \geq n,\, m \in \mathbb{N} \big\rangle \tag{A.6}
$$

generated by all products of Virasoro modes whose L_0 grading is greater than or equal to n where $n \geq 0$. This subalgebra can be used to define a filtration of the Virasoro module $\mathcal{H}$ as the family

$$
\mathcal{H}^n = \mathcal{H}/\mathcal{A}_{n+1}\mathcal{H} \tag{A.7}
$$

of quotient spaces. Here we refer to the integer n as the Nahm level. At Nahm level $n \in \mathbb{N}_0$, the co-multiplications $\Delta(L_{-m})$ and $\tilde{\Delta}(L_{-m})$ vanish for $m > n$. At level 0, all co-multiplications of negative Virasoro modes thus vanish. It follows, in particular, that

$$L_{-1}^{i} \times L_{-1}^{j} = (-1)^{j} L_{-1}^{i+j} \times I \tag{A.8}$$

at level 0. Furthermore, from

$$\Delta(L_0)\big\{L_{-1}^{\ell} \times I\big\}|\Delta\rangle \times |\Delta'\rangle = \big\{L_{-1}^{\ell+1} \times I + \big(\Delta + \Delta' + \ell\big)L_{-1}^{\ell} \times I\big\}|\Delta\rangle \times |\Delta'\rangle \tag{A.9}$$

it follows by induction that

$$0 = \big\{L_{-1}^{k} \times I\big\}|\Delta\rangle \times |\Delta'\rangle \quad \Rightarrow \quad 0 = \big\{L_{-1}^{\ell} \times I\big\}|\Delta\rangle \times |\Delta'\rangle, \quad \ell \geq k \tag{A.10}$$

Let us consider the fusion product $(r,1) \otimes (1,s)$ assuming that $(1,s)$ is the highest-weight module $Q_{1,s}$ while recalling that $(r,1) = Q_{r,1}$. As an *initial* vector space for the examination of this fusion product at Nahm level 0, we may consider

$$\Big\{\big\{L_{-1}^{\ell} \times I\big\}|\Delta_{r,1}\rangle \times |\Delta_{1,s}\rangle;\ \ell \in \mathbb{Z}_{0,r-1}\Big\} \tag{A.11}$$

or the similar set based on s vectors of the form $\{I \times L_{-1}^{\ell}\}|\Delta_{r,1}\rangle \times |\Delta_{1,s}\rangle$. Indeed, we can use the vanishing of the singular vector $|\lambda_{r,1}\rangle$ in

$$|\lambda_{r,1}\rangle \times |\Delta_{1,s}\rangle \tag{A.12}$$

to express the vector $\{L_{-1}^{r} \times I\}|\Delta_{r,1}\rangle \times |\Delta_{1,s}\rangle$ first in terms of vectors of the form $\{(L_{-n_1} \ldots L_{-n_j}) \times I\}|\Delta_{r,1}\rangle \times |\Delta_{1,s}\rangle$ where $j < r$ and subsequently in terms of the initial vectors (A.11)

$$\big\{L_{-1}^{r} \times I\big\}|\Delta_{r,1}\rangle \times |\Delta_{1,s}\rangle = \Big\{\sum_{\ell=0}^{r-1} \alpha_{\ell}^{r} L_{-1}^{\ell} \times I\Big\}|\Delta_{r,1}\rangle \times |\Delta_{1,s}\rangle \tag{A.13}$$

The relation (A.9) then allows us to ignore $\{L_{-1}^{k} \times I\}|\Delta_{r,1}\rangle \times |\Delta_{1,s}\rangle$ for $k > r$. For $r = 2$, in particular, the decomposition (A.13) reads

$$L_{-1}^{2} \times I = -pL_{-1} \times I + p\Delta_{1,s} I \times I \tag{A.14}$$

The initial vector space $Q_{r,1}^{s} \otimes Q_{1,s}^{n}$ at Nahm level n is constructed as the tensor product of the "special subspace" $Q_{r,1}^{s} = \{L_{-1}^{\ell}|\Delta_{r,1}\rangle;\ \ell \in \mathbb{Z}_{0,r-1}\}$ of $Q_{r,1}$ and the space $Q_{1,s}^{n} = \{L_{-k_1} \ldots L_{-k_m}|\Delta_{1,s}\rangle;\ k_1 + \ldots + k_m \leq n\}$ of states up to Virasoro level n in $Q_{1,s}$. Depending on the model (labelled by p) and the second fusion factor $(1,s)$, the linear span of the initial vector space may contain a spurious subspace [9] generated by linear relations on the initial vector space. The fusion space $(Q_{r,1} \otimes Q_{1,s})_{\mathrm{f}}^{n}$ at Nahm level n is the complement to the spurious subspace of $Q_{r,1}^{s} \otimes Q_{1,s}^{n}$.

A.4 Kac representation $(2,3)$ in critical dense polymers $\mathcal{LM}(1,2)$

We consider the fusion $(2,1) \otimes (1,3) = (2,3)$ for $p = 2$. The Kac representation $(2,1)$ is the irreducible highest-weight module $Q_{2,1} = M_{2,1} = \mathcal{V}(1)$ generated from the highest-weight vector $|\Delta_{2,1}\rangle$ where

$$|\lambda_{2,1}\rangle = \big(L_{-1}^{2} - 2L_{-2}\big)|\Delta_{2,1}\rangle = 0, \qquad \Delta_{2,1} = 1 \tag{A.15}$$

The Kac representation $(1,3)$ is the reducible yet indecomposable highest-weight module $Q_{1,3}$ generated from the highest-weight vector $|\Delta_{1,3}\rangle$ where

$$|\lambda_{1,3}\rangle = \big(L_{-1}^3 - 2L_{-2}L_{-1}\big)|\Delta_{1,3}\rangle = 0, \qquad \Delta_{1,3} = 0 \tag{A.16}$$

Here we analyze the fusion $(2,1)\otimes(1,3) = (2,3)$ up to Nahm level 2 and find that the eight-dimensional space $Q_{2,1}^s \otimes Q_{1,3}^2$ containing the fusion space $\big(Q_{2,1}\otimes Q_{1,3}\big)_{\mathrm{f}}^2$ of our interest also contains a two-dimensional spurious subspace defined by the relations

$$\begin{aligned} 0 &= \big\{L_{-1}\times L_{-1} + I\times L_{-1}^2\big\}|\Delta_{2,1}\rangle\otimes|\Delta_{1,3}\rangle \\ 0 &= \big\{2I\times L_{-1} + 2L_{-1}\times L_{-1} + 4I\times L_{-2} - L_{-1}\times L_{-1}^2 + 2L_{-1}\times L_{-2}\big\}|\Delta_{2,1}\rangle\otimes|\Delta_{1,3}\rangle \end{aligned} \tag{A.17}$$

Likewise, we find

$$\begin{aligned} \big(Q_{2,1}\otimes Q_{1,3}\big)_{\mathrm{f}}^1 &= \big(Q_{2,1}^s\otimes Q_{1,3}^1\big)/\big(0 = 2|\Delta_{2,1}\rangle\otimes L_{-1}|\Delta_{1,3}\rangle + L_{-1}|\Delta_{2,1}\rangle\otimes L_{-1}|\Delta_{1,3}\rangle\big) \\ \big(Q_{2,1}\otimes Q_{1,3}\big)_{\mathrm{f}}^0 &= Q_{2,1}^s\otimes Q_{1,3}^0 \end{aligned} \tag{A.18}$$

The Virasoro generator $\Delta(L_0)$ is diagonalizable on the spaces $\big(Q_{2,1}\otimes Q_{1,3}\big)_{\mathrm{f}}^n$, $n = 0,1,2$, with suitably normalized eigenvectors given by

$$\begin{aligned} |0\rangle^0 &= 3L_{-1}|\Delta_{2,1}\rangle\otimes|\Delta_{1,3}\rangle \\ |1\rangle^0 &= \big\{I\times I + L_{-1}\times I\big\}|\Delta_{2,1}\rangle\otimes|\Delta_{1,3}\rangle \end{aligned} \tag{A.19}$$

$$\begin{aligned} |0\rangle^1 &= -3|\Delta_{2,1}\rangle\otimes L_{-1}|\Delta_{1,3}\rangle \\ |1\rangle^1 &= \big\{I\times I - L_{-1}\times I - 2I\times L_{-1}\big\}|\Delta_{2,1}\rangle\otimes|\Delta_{1,3}\rangle \\ |2\rangle^1 &= \big\{L_{-1}\times I + I\times L_{-1}\big\}|\Delta_{2,1}\rangle\otimes|\Delta_{1,3}\rangle \end{aligned} \tag{A.20}$$

and

$$\begin{aligned} |0\rangle^2 &= \big\{I\times L_{-1} + L_{-1}\times L_{-1} + 2I\times L_{-2} + L_{-1}\times L_{-2}\big\}|\Delta_{2,1}\rangle\otimes|\Delta_{1,3}\rangle \\ |1\rangle^2 &= \big\{2I\times L_{-1} + L_{-1}\times L_{-1} + 3I\times L_{-2} + L_{-1}\times L_{-2}\big\}|\Delta_{2,1}\rangle\otimes|\Delta_{1,3}\rangle \\ |2\rangle_1^2 &= \big\{2I\times L_{-1} + 2I\times L_{-2} + L_{-1}\times L_{-2}\big\}|\Delta_{2,1}\rangle\otimes|\Delta_{1,3}\rangle \\ |2\rangle_2^2 &= \big\{2I\times I - L_{-1}\times I - I\times L_{-1} - L_{-1}\times L_{-1} - 2I\times L_{-2}\big\}|\Delta_{2,1}\rangle\otimes|\Delta_{1,3}\rangle \\ |3\rangle_1^2 &= \big\{-I\times I + L_{-1}\times I + 2I\times L_{-1} + 3I\times L_{-2} + L_{-1}\times L_{-2}\big\}|\Delta_{2,1}\rangle\otimes|\Delta_{1,3}\rangle \\ |3\rangle_2^2 &= \big\{-2I\times I + 2L_{-1}\times I + 2I\times L_{-1} + L_{-1}\times L_{-1} + 2I\times L_{-2}\big\}|\Delta_{2,1}\rangle\otimes|\Delta_{1,3}\rangle \end{aligned} \tag{A.21}$$

When restricting from level 2 to level 1, it is $|2\rangle_2^2$ which becomes $|2\rangle^1$, while the somewhat unusual normalizations of $|0\rangle^0$ and $|0\rangle^1$ follow from the normalization of $|0\rangle^2$ under similar level restrictions. The actions of the Virasoro generators $L_{\pm1}$ and $L_{\pm2}$ on these states are also examined using the co-multiplication and we find that the non-trivial actions read

$$\begin{gathered} L_{-2}|0\rangle = 3|2\rangle_1, \qquad L_2|2\rangle_1 = -\tfrac{1}{3}|0\rangle, \qquad L_1|1\rangle = -\tfrac{1}{3}|0\rangle, \qquad L_2|2\rangle_2 = -|0\rangle \\ L_{-1}|1\rangle = |2\rangle_2, \qquad L_{-1}|2\rangle_2 = |3\rangle_2, \qquad L_{-2}|1\rangle = |3\rangle_1 \\ L_1|2\rangle_2 = 2|1\rangle, \qquad L_1|3\rangle_1 = 3|2\rangle_2, \qquad L_1|3\rangle_2 = 6|2\rangle_2, \qquad L_2|3\rangle_1 = 3|1\rangle, \qquad L_2|3\rangle_2 = 6|1\rangle \end{gathered} \tag{A.22}$$

where the level indications have been omitted. With

$$\begin{array}{rcl}\{|0\rangle,|2\rangle_1\} & \subset & \mathcal{V}(0)\\ \{|1\rangle,|2\rangle_2,|3\rangle_1,|3\rangle_2\}/\big(0=|3\rangle_2-2|3\rangle_1\big) & \subset & \mathcal{V}(1)\\ \{|3\rangle_2-2|3\rangle_1\} & \subset & \mathcal{V}(3)\end{array} \tag{A.23}$$

this is in accordance with the conjectured structure (2.34) of the non-highest-weight module $(2,3)$ appearing in the fusion $(2,1)\otimes(1,3)=(2,3)$. It is noted that

$$|3\rangle_{\rm sing}=|3\rangle_2-2|3\rangle_1 \tag{A.24}$$

corresponds to the singular vector at Virasoro level 2 in $(1,5)$ from which the submodule $\mathcal{V}(3)$ is generated, and that the reducible yet indecomposable highest-weight module $(1,5)$ admits the short exact sequence

$$0\to\mathcal{V}(3)\to(1,5)\to\mathcal{V}(1)\to 0 \tag{A.25}$$

A.5 Fusion product $(1,3)\otimes(1,3)$ in critical dense polymers $\mathcal{LM}(1,2)$

Here we analyze the fusion product $(1,3)\otimes(1,3)$ up to Nahm level 2 in $\mathcal{LM}(1,2)$. We use the singular vector (A.16) and find that the twelve-dimensional space $Q_{1,3}^s\otimes Q_{1,3}^2$ containing the fusion space $\big(Q_{1,3}\otimes Q_{1,3}\big)_{\rm f}^2$ also contains a two-dimensional spurious subspace defined by the relations

$$\begin{array}{rcl}0 & = & \big\{2L_{-1}\times L_{-1}-2L_{-1}\times L_{-1}^2-L_{-1}^2\times L_{-1}^2+L_{-1}\times L_{-2}+2L_{-1}^2\times L_{-2}\big\}|\Delta_{1,3}\rangle\otimes|\Delta_{1,3}\rangle\\ 0 & = & \big\{L_{-1}^2\times L_{-1}+L_{-1}\times L_{-1}^2\big\}|\Delta_{1,3}\rangle\otimes|\Delta_{1,3}\rangle\end{array} \tag{A.26}$$

Likewise, we find

$$\begin{array}{rcl}\big(Q_{1,3}\otimes Q_{1,3}\big)_{\rm f}^1 & = & \big(Q_{1,3}^s\otimes Q_{1,3}^1\big)/\big(0=2L_{-1}|\Delta_{1,3}\rangle\otimes L_{-1}|\Delta_{1,3}\rangle+L_{-1}^2|\Delta_{1,3}\rangle\otimes L_{-1}|\Delta_{1,3}\rangle\big)\\ \big(Q_{1,3}\otimes Q_{1,3}\big)_{\rm f}^0 & = & Q_{1,3}^s\otimes Q_{1,3}^0\end{array} \tag{A.27}$$

The conjectured decomposition of the fusion product $(1,3)\otimes(1,3)$ is described by the structure diagram

$$\mathcal{R}_1^1\oplus(1,5):\qquad \begin{array}{ccc} & \mathcal{V}(1) & \\ \swarrow & & \nwarrow\\ \mathcal{V}(0)_0 & \longleftarrow & \mathcal{V}(0)_1\end{array}\qquad\bigoplus\qquad \begin{array}{cc}\tilde{\mathcal{V}}(3) & \\ & \nwarrow\\ & \tilde{\mathcal{V}}(1)\end{array} \tag{A.28}$$

The indices on the irreducible highest-weight modules $\mathcal{V}(0)$ and the tildes on the irreducible subfactors of $(1,5)$ are immaterial but introduced for ease of reference to the different modules. The rank-2 module $\mathcal{R}_1^1$ is generated by the action of the Virasoro modes on the vectors $|0\rangle_0$ and $|0\rangle_1$ where

$$L_0|0\rangle_1=|0\rangle_0 \tag{A.29}$$

while the highest-weight module $(1,5)$ is generated from the vector $|\tilde{1}\rangle$. At Nahm level 2, we should then recover

$$\begin{array}{rcl}\{|0\rangle_0,L_{-2}|0\rangle_0\} & \subset & \mathcal{V}(0)_0\\ \{|0\rangle_1,L_{-2}|0\rangle_1\} & \subset & \mathcal{V}(0)_1\\ \{L_{-1}|0\rangle_1,L_{-1}^2|0\rangle_1\} & \subset & \mathcal{V}(1)\end{array} \tag{A.30}$$

and

$$
\begin{aligned}
\{|\tilde{1}\rangle, L_{-1}|\tilde{1}\rangle, L_{-1}^2|\tilde{1}\rangle, L_{-2}|\tilde{1}\rangle\}/\big(0=(L_{-1}^2-2L_{-2})|\tilde{1}\rangle\big) &\subset \tilde{\mathcal{V}}(1)\\
\{(L_{-1}^2-2L_{-2})|\tilde{1}\rangle\} &\subset \tilde{\mathcal{V}}(3)
\end{aligned}
\tag{A.31}
$$

satisfying

$$
L_0L_{-2}|0\rangle_1 = 2L_{-2}|0\rangle_1 + L_{-2}|0\rangle_0, \qquad L_2(L_{-1}^2-2L_{-2})|\tilde{1}\rangle = 0 \tag{A.32}
$$

in particular. This is confirmed when considering

$$
\begin{aligned}
|0\rangle_0 &= \big(0,0,0,0,1,0,-1,0,2,1\big)\\
L_{-2}|0\rangle_0 &= \big(0,0,0,0,6,0,0,0,6,3\big)\\
|0\rangle_1 &= \big(-3,2,-\tfrac{1}{2},0,1,0,0,0,1,0\big) + \alpha|0\rangle_0\\
L_{-2}|0\rangle_1 &= \big(0,-6,3,0,-5,0,0,-3,-2,-\tfrac{5}{2}\big) + \alpha L_{-2}|0\rangle_0\\
L_{-1}|0\rangle_1 &= \big(0,0,0,-3,0,0,\tfrac{3}{2},0,-3,0\big)\\
L_{-1}^2|0\rangle_1 &= \big(0,-6,3,0,0,-3,-3,0,6,0\big)
\end{aligned}
\tag{A.33}
$$

and

$$
\begin{aligned}
|\tilde{1}\rangle &= \big(0,0,0,-2,8,0,-3,0,10,4\big)\\
L_{-1}|\tilde{1}\rangle &= \big(0,4,-2,0,-4,-2,2,0,-4,0\big)\\
L_{-1}^2|\tilde{1}\rangle &= \big(0,-8,8,0,8,0,-4,0,8,0\big)\\
L_{-2}|\tilde{1}\rangle &= \big(0,-4,4,0,8,0,0,0,12,4\big)
\end{aligned}
\tag{A.34}
$$

in the ordered basis

$$
\begin{gathered}
\big\{I\times I,\, L_{-1}\times I,\, L_{-1}^2\times I,\, I\times L_{-1},\, L_{-1}\times L_{-1},\, I\times L_{-1}^2,\, L_{-1}\times L_{-1}^2,\\
I\times L_{-2},\, L_{-1}\times L_{-2},\, L_{-1}^2\times L_{-2}\big\}|\Delta_{1,3}\rangle\otimes|\Delta_{1,3}\rangle
\end{gathered}
\tag{A.35}
$$

The parameter α in (A.33) is free and corresponds to a gauge transformation.

B Kac fusion algebra for critical dense polymers $\mathcal{LM}(1,2)$

The Kac fusion algebra for critical dense polymers $\mathcal{LM}(1,2)$ satisfies

$$
\big\langle (r,s);\, r,s\in\mathbb{N}\big\rangle = \big\langle (r,2),\, (r,2j-1),\, \mathcal{R}_r;\, r,j\in\mathbb{N}\big\rangle \tag{B.1}
$$

where $\mathcal{R}_r = \mathcal{R}_r^1$. The fusion rules are

$$\begin{aligned}
(r,2)\otimes(r',2) &= \bigoplus_{\ell=|r-r'|+1,\,\mathrm{by}\,2}^{r+r'-1} \mathcal{R}_\ell \\
(r,2)\otimes(r',2j'-1) &= \bigoplus_{\ell=|r-r'|+1,\,\mathrm{by}\,2}^{r+r'-1}\ \bigoplus_{k=|\ell-j'+\frac{1}{2}|+\frac{1}{2}}^{\ell+j'-1} (k,2) \\
(r,2)\otimes\mathcal{R}_{r'} &= \bigoplus_{\ell=|r-r'|}^{r+r'} \big(2-\delta_{\ell,|r-r'|}-\delta_{\ell,r+r'}\big)(\ell,2) \\
(r,2j-1)\otimes(r',2j'-1) &= \bigoplus_{\ell=|r-r'|+1,\,\mathrm{by}\,2}^{r+r'-1}\ \bigoplus_{k=|j-j'|+1,\,\mathrm{by}\,2}^{j+j'-3}\ \bigoplus_{i=|\ell-k|+1,\,\mathrm{by}\,2}^{\ell+k-1} \mathcal{R}_i \oplus \bigoplus_{\ell=|r-r'|+1,\,\mathrm{by}\,2}^{r+r'-1} (\ell,2j+2j'-3) \\
(r,2j-1)\otimes\mathcal{R}_{r'} &= \bigoplus_{\ell=|r-r'|+1,\,\mathrm{by}\,2}^{r+r'-1}\ \bigoplus_{k=|\ell-j+\frac{1}{2}|+\frac{1}{2}}^{\ell+j-1} \mathcal{R}_k \\
\mathcal{R}_r\otimes\mathcal{R}_{r'} &= \bigoplus_{\ell=|r-r'|}^{r+r'} \big(2-\delta_{\ell,|r-r'|}-\delta_{\ell,r+r'}\big)\mathcal{R}_\ell
\end{aligned} \tag{B.2}$$

where it is noted that some summations are in steps of 1 while others are in steps of 2. The vertical fusion rule

$$(1,2j-1)\otimes(1,2j'-1) = \bigoplus_{\ell=|j-j'|+1,\,\mathrm{by}\,2}^{j+j'-3} \mathcal{R}_\ell \oplus (1,2j+2j'-3) \tag{B.3}$$

revises the similar formula in [28].

C Jordan canonical form of the hamiltonian $H_{s,s'}^{(N)}$

Let us introduce the $d\times d$ square matrix $D = D^{(d)}$

$$D = \begin{pmatrix} 0 & 1 & & & & & \\ 1 & 0 & 1 & & & & \\ & 1 & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{pmatrix} \tag{C.1}$$

It has d eigenvalues

$$\alpha_j = 2\cos\frac{j\pi}{d+1}, \qquad j\in\mathbb{Z}_{1,d} \tag{C.2}$$

all of which are distinct. The associated eigenvectors can be normalized as

$$v_j = \begin{pmatrix} \sin \frac{j\pi}{d+1} \\ \vdots \\ \sin \frac{ij\pi}{d+1} \\ \vdots \\ \sin \frac{dj\pi}{d+1} \end{pmatrix}, \qquad i,j \in \mathbb{Z}_{1,d} \tag{C.3}$$

The matrix D is diagonalized by $K = K^{(d)}$ constructed by concatenating the eigenvectors (C.3). That is,

$$K^{-1}DK = \mathrm{diag}(\alpha_1, \ldots, \alpha_d) \tag{C.4}$$

where

$$K_{ij} = \sqrt{\frac{2}{d+1}} \sin \frac{ij\pi}{d+1}, \qquad i,j \in \mathbb{Z}_{1,d} \tag{C.5}$$

satisfying

$$K^{-1} = K^t = K \tag{C.6}$$

We can construct the hamiltonian $H_{s,s'}^{(N)}$ discussed in Section 3.2.2 as the block matrix

$$-H_{s,s'}^{(N)} = \begin{pmatrix} D_\beta & \delta^{(N-1)} \\ 0_{1\times(N-1)} & 0 \end{pmatrix} \tag{C.7}$$

where $D_\beta = D_\beta^{(N-1)}$ is the $(N-1)\times(N-1)$ square matrix

$$D_\beta = D + \beta I, \qquad \beta = -2\cos\tfrac{\pi}{p} \tag{C.8}$$

while the entries of the $(N-1)$-vector $\delta^{(N-1)}$ are

$$\delta_j^{(N-1)} = \delta_{j,s-t-1}, \qquad j \in \mathbb{Z}_{1,N-1} \tag{C.9}$$

The matrix $-H_{s,s'}^{(N)}$ is similar to

$$-\bar{H}_{s,s'}^{(N)} = -\bar{K}^{-1} H_{s,s'}^{(N)} \bar{K}, \qquad \bar{K} = \begin{pmatrix} K & 0_{(N-1)\times 1} \\ 0_{1\times(N-1)} & 1 \end{pmatrix} \tag{C.10}$$

that is,

$$-\bar{H}_{s,s'}^{(N)} = \begin{pmatrix} \beta + 2\cos\frac{\pi}{N} & 0 & \ldots & \ldots & 0 & \sqrt{\frac{2}{N}}\sin\frac{(s-t-1)\pi}{N} \\ 0 & \ddots & & & \vdots & \vdots \\ \vdots & & \beta + 2\cos\frac{j\pi}{N} & & \vdots & \sqrt{\frac{2}{N}}\sin\frac{j(s-t-1)\pi}{N} \\ \vdots & & & \ddots & 0 & \vdots \\ \vdots & & & & \beta + 2\cos\frac{(N-1)\pi}{N} & \sqrt{\frac{2}{N}}\sin\frac{(N-1)(s-t-1)\pi}{N} \\ 0 & \ldots & \ldots & \ldots & 0 & 0 \end{pmatrix} \tag{C.11}$$

Since two similar matrices have the same Jordan canonical form, this simple result facilitates a straightforward analysis of the Jordan decomposition of $H_{s,s'}^{(N)}$ itself. From $\beta + 2\cos\frac{j\pi}{N} \neq \beta + 2\cos\frac{j'\pi}{N}$ for $j \neq j'$, it follows that $H_{s,s'}^{(N)}$ is diagonalizable if $\beta \neq -2\cos\frac{j\pi}{N}$ for all $j \in \mathbb{Z}_{1,N-1}$. It is also readily seen that $H_{s,s'}^{(N)}$ is diagonalizable if $\beta + 2\cos\frac{j_0\pi}{N} = \sin\frac{j_0(s-t-1)\pi}{N} = 0$ for some $j_0 \in \mathbb{Z}_{1,N-1}$, while $H_{s,s'}^{(N)}$ is non-diagonalizable if there exists $j_0 \in \mathbb{Z}_{1,N-1}$ for which $\beta = -2\cos\frac{j_0\pi}{N}$ and $\sin\frac{j_0(s-t-1)\pi}{N} \neq 0$.

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